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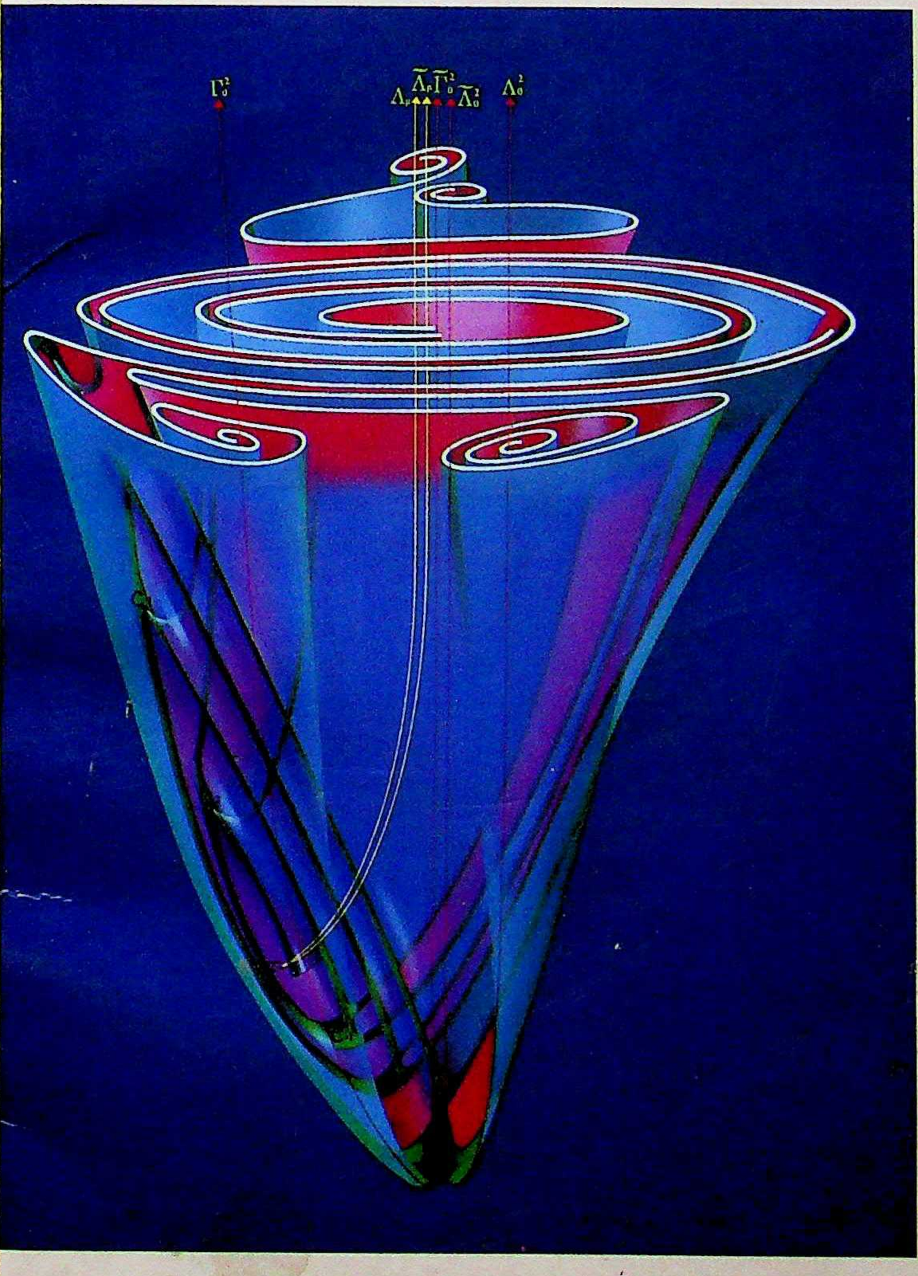
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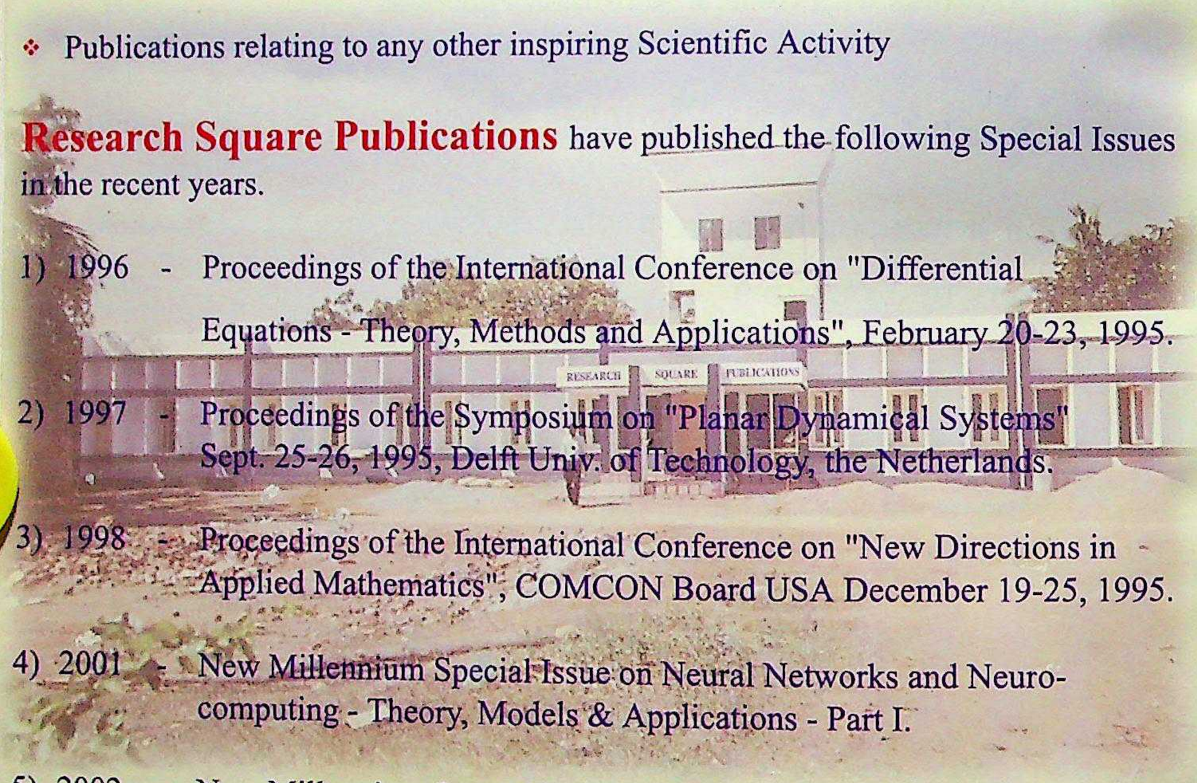

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Existence of Periodic Solutions for Quasilinear Ordinary Differential Equations with Discontinuities

Nikolaos Matzakos and Nikolaos S. Papageorgiou

(Submitted by: V. Sree Hari Rao)

131017

Abstract

We consider a quasilinear differential equation with discontinuous right hand side and periodic boundary conditions. To obtain an existence theory we pass to a relevant multivalued variant of the original problem, which we solve. Our approach is a mixture of the variational method (for nonsmooth locally Lipschitz functionals) and of the method of upper and lower solutions. The mixing of these two techniques is made possible by a nonresonance condition below the first nonzero eigenvalue of the one-dimensional p–Laplacian with periodic boundary conditions.

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1 Introduction

In this paper we study the following periodic problem for quasilinear differential equations

$$\left\{\begin{array}{l} -(|x^{'}(t)|^{p-2}x^{'}(t))^{'} = f(t,x(t)) \ \text{ a.e. on } T=[0,b] \\ x(0)=x(b),\ x^{'}(0)=x^{'}(b),\ a \leq p < \infty \end{array}\right\} \tag{1}$$

We do not assume $f(t,\cdot)$ to be continuous. So problem (1) need not have a solution. In order to be able to develop a satisfactory existence theory, we need to pass to a multivalued problem, which is obtained by, roughly speaking, filling in the gaps at the discontinuity points of $f(t,\cdot)$. For this purpose we introduce the following two functions:

$$f_0(t,x) = \lim_{\varepsilon\downarrow 0} ess \inf_{|x-x^{'}|\leq\varepsilon} f(t,x^{'})$$

$$\text{and } f_1(t,x) = \sup_{\varepsilon\downarrow 0} ess \inf_{|x-x^{'}|\leq\varepsilon} f(t,x^{'})$$

Evidently $f_0(t,x) \leq f_1(t,x)$ for all $(t,x) \in T \times \mathbf{R}$. Then instead of (1), we consider the following multivalued version of it:

$$\left\{\begin{array}{l} -(|x^{'}(t)|^{p-2}x^{'}(t))^{'} \in [f_0(t,x(t)), f_1(t,x(t))] \ \text{ a.e. on } T \\ x(0)=x(b),\ x^{'}(0)=x^{'}(b),\ a \leq p < \infty \end{array}\right\} \tag{2}$$

We prove the existence of a solution for problem (2) by combining the variational approach with the method of upper and lower solutions. The link between these two techniques is a nonresonance condition below the first nonzero eigenvalue of the one dimensional p–Laplacian with periodic boundary conditions.

Quasilinear ordinary differential equations with Dirichlet boundary conditions were studied by Boccardo–Drabek–Giachetti–Kucera [1], Drabek [7], DelPino–Elgueta–Manasevich [6] and De Coster [5]. Under homogeneous Dirichlet boundary conditions, the differential operator $Ax = -(|x^{'}|^{p-2}x^{'})^{'}$ (the one dimensional p–Laplacian) is invertible (as a nonlinear map between appropriate spaces) and so classical Leray–Schauder degree techniques (like the Leray–Schauder principle) can be used. In contrast for the periodic problem, the corresponding operator has a nontrivial kernel and so a different aopproach is needed. This problem was examined by Guo [9], who considered a more general version of (1) by allowing f to depend also on $x^{'}$. However he assumed f to be continuous in all three variables (including the time–variable). His approach was degree theoretic, based on Mawhin's coincidence degree theory. This forced him to introduce additional restrictive conditions on the function f (like growth condition, see hypothesis H_1, p. 710 in Guo [9]). Our method of proof here is completely different from that of Guo and is based on variational arguments mixed with ideas from the method of upper and lower solutions.

2 Preliminaries

Since the function $f(t,\cdot)$ is discontinuous, our variational technique will be based on the critical point theory for nonsmooth locally Lipschitz energy functionals. For easy reference we recall here the basic aspects of that theory. For details we refer to Chang [3].

Let X be a Banach space and $f : X \to \mathbf{R}$ a function. We say that $f(\cdot)$ is locally Lipschitz, if for every $x \in X$, there is a neighborhood U of x and a $k \geq 0$ depending on U such that

$$|f(y) - f(z)| \leq k \,\|y - z\|$$

for all $y, z \in U$. For every $h \in X$, we define the generalized directional derivative $f^o(x;h)$ by

$$f^o(x;h) = \overline{\lim_{\substack{x^{'} \to x \\ \lambda \downarrow 0}}} \frac{f(x^{'} + \lambda h) - f(x^{'})}{\lambda}.$$

It is easy to check that $h \to f^o(x;h)$ is sublinear, $|f^o(x;h)| \leq k\,\|h\|$ and $h \to f^o(x;h)$ is continuous. So $f^o(x;\cdot)$ is the support function of a nonempty, convex and w^*–compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^o(x;h) \ \text{ for all } \ h \in X\}$$

known as the "generalized subdifferential of $f(\cdot)$ at x" (see Clarke [4]).

For every $x^* \in \partial f(x)$, we have $\|x^*\| \le k$. Also if $f, g : X \to \mathbf{R}$ are locally Lipschitz functions, then $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbf{R}$. If $f : X \to \mathbf{R}$ is convex, then it is well-known from convex analysis that $f(\cdot)$ is locally Lipschitz and its subdifferential in the sense of convex analysis coincides with the generalized subdifferential introduced above.

Let $f : X \to \mathbf{R}$ be locally Lipschitz on the Banach space x. A point X is said to be a "critical point" of $f(\cdot)$ if $0 \in \partial f(x)$. It is easy to see that if $x \in X$ is a local mininmum of $f(\cdot)$, then x is a critical point of $f(\cdot)$. We say that $f(\cdot)$ satisfies the "(PS)-condition" (Palais-Smale condition), if any sequence $\{x_n\}_{n\ge1} \subseteq X$ along which $\{f(x_n)\}_{n\ge1}$ is bounded and $m(x_n) = \inf\{\|x^*\| : x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. Since for $f \in C^1(X, \mathbf{R})$, $\partial f(x) = \{f'(x)\}$, we see that for a smooth $f(\cdot)$, the above version of the (PS)-condition coincides with the classical one (see Rabinowitz [12]).

In our analysis of problem (2), we will need the following theorem, which is due to Chang [3] and extends to a nonsmooth setting the well-known saddle point theorem of Rabinowitz [12]. **Theorem 1.** If X is a reflexive Banach space, $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$, $f : X \to \mathbf{R}$ is a locally Lipschitz function which satisfies the (PS)-condition and there exist constants $\beta_1 < \beta_2$ and a neighborhood U of 0 in X_1 such that $f|_{\partial U} \le \beta_1$ and $f|_{X_2} \ge \beta_2$, then $f(\cdot)$ has a critical point x and $f(x) = c \ge \beta_2$.

3 Auxiliary Results

In this section we prove two auxiliary results which will allow us in Section 4 to use the method of upper and lower solutions together with theorem 1 in order to obtain a nontrivial solution for problem (2).

First let us define what we mean by a solution of problem (2).

Definition. By a solution of (2) we mean a function $x \in C^1(T)$ such that $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)(\frac{1}{p} + \frac{1}{q} = 1)$, there exists $g \in L^q(T)$ such that $f_o(t, x(t)) \le g(t) \le f_1(t, x(t))$ a.e. on T, $-(|x'(t)|^{p-2}x'(t))' = g(t)$ a.e. on T and $x(0) = x(b)$, $x'(0) = x'(b)$.

In what follows $W^{1,p}_{per}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$. Recall that

$W^{1,p}(T)$ is continuously embedded in $C(T)$ and so the pointwise evaluations at $t = 0$ and $t = b$ make sense. Let $\lambda_2 > 0$ be the second eigenvalue of the one–dimensional p–Laplacian with periodic boundary conditions, i.e.

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = \lambda_2|x(t)|^{p-2}x(t) \ \text{ a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array}\right\} \tag{3}$$

Notice that $\lambda_1 = 0$ is the first eigenvalue. So $\lambda_2 > 0$ is the first nonzero eigenvalue of the p–Laplacian with periodic boundary conditions (see Showalter [13], Corollary 7.D, p. 78).
Now we introduce our hypothesis for the function $f(t,x)$. They will be in effect for the rest of this paper. Recall that a function $h : T \times \mathbf{R} \to \mathbf{R}$ is said to be "N–measurable", if for every $x : T \to \mathbf{R}$ Borel measurable, we have that $t \to h(t, x(t))$ is Borel measurable (superpositional measurability).

H(F): $f : T \times \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that

(i) f_0, f_1 are N–measurable functions;

(ii) for every $M > 0$, there exists $\gamma_M \in L^q(T)$ such that for almost all $t \in T$ and all $|x| \le M$, $|f(t,x)| \le \gamma_M(t)$;

(iii) $\displaystyle\overline{\lim_{|x|\to\infty}} \frac{f(t,x)}{|x|^{p-2}x} < \lambda_2$ uniformly for almost all $t \in T$,

(iv) for almost all $t \in T$ and all $x \in \mathbf{R} \setminus \{0\}$ $f(t,x)x > 0$ (strict sign condition).

By virtue of hypothesis $H(F)(iii)$, we see that there exists $\varepsilon > 0$ (with $0 < \lambda_2 - \varepsilon$) and $M > 0$ such that

$$f(t,x) \le (\lambda_2 - \varepsilon)|x|^{p-2}x \text{for almost all } t \in T \text{ and all } x \ge M > 0, \tag{4}$$

and

$$f(t,x) \ge (\lambda_2 - \varepsilon)|x|^{p-2}x \text{for almost all } t \in T \text{ and all } x \le M < 0. \tag{5}$$

Moreover, from hypothesis $H(F)(ii)$ we have that

$$|f(t,x)| \le \gamma_M(t) \text{ for almost all } t \in T \text{ and all } |x| \le M \tag{6}$$

Combining (4), (5) and (6) above, we infer that for some $\varepsilon > 0$ (with $0 < \lambda_2 - \varepsilon$) and $\gamma \in L^q(T)$ $\gamma \geq 0$, we have

$$f(t,x) \leq (\lambda_2 - \varepsilon)|x|^{p-2}x + \gamma(t) \text{for almost all } t \in T \text{ and all } x \geq 0$$

and

$$f(t,x) \geq (\lambda_2 - \varepsilon)|x|^{p-2}x - \gamma(t) \text{for almost all } t \in T \text{ and all } x \leq 0.$$

We consider the following two auxiliary periodic problems:

$$\left\{ \begin{array}{l} -(|\phi'(t)|^{p-2}\phi'(t))' = (\lambda_2 - \varepsilon)|\phi(t)|^{p-2}\phi(t) + \gamma(t) \ \text{ a.e. on } T \\ \phi(0) = \phi(b), \ \phi'(0) = \phi'(b) \end{array} \right\} \quad (7)$$

and

$$\left\{ \begin{array}{l} -(|\psi'(t)|^{p-2}\psi'(t))' = (\lambda_2 - \varepsilon)|\psi(t)|^{p-2}\psi(t) - \gamma(t) \ \text{ a.e. on } T \\ \psi(0) = \psi(b), \ \psi'(0) = \psi'(b). \end{array} \right\} \quad (8)$$

By a solution of problem (7) (resp. of problem (8)), we mean a function $\phi \in C^1(T)$ (resp. $\psi \in C^1(T)$) with $|\phi'|^{p-2}\phi' \in W^{1,q}(T)$ (resp. $|\psi'|^{p-2}\psi' \in W^{1,q}(T)$), which satisfies (7) (resp.(8)). In the next two propositions we prove that the two problems have solutions $\phi \geq 0$ and $\psi \leq 0$ respectively.

Proposition 2. Problem (7) has a solution $\phi \geq 0$, $\phi \neq 0$.

Proof. Since $\gamma(t) \geq 0$ a.e. on T, from Theorem 9.5, p. 210 of Gilbarg–Trudinger [8], we know that if $\phi \in C^1(T)$ is a solution of (7), then $\phi(t) \geq 0$ for all $t \in T$.

So we have to show that a solution ϕ exists. To this end let $W^{1,p}_{per}(T) = Z \oplus Y$, where Z is the space of constant functions (i.e. $Z = \mathbf{R}$) and Y is the space of all functions in $W^{1,p}_{per}(T)$ with mean value zero (i.e. $Y = \{y \in W^{1,p}_{per}(T) : \int_0^b y(t)\,dt = 0\}$). Let $R : W^{1,p}_{per}(T) \to \mathbf{R}$ be defined by

$$R(x) = \frac{1}{p}\left\|x'\right\|_p^p - \frac{(\lambda_2 - \varepsilon)}{p}\left\|x\right\|_p^p - \int_0^b \gamma(t)x(t)\,dt$$

If $z \in Z = \mathbf{R}$, then we have

$$R(z) = -\frac{(\lambda_2 - \varepsilon)}{p}\left\|z\right\|_p^p - z\int_0^b \gamma(t)\,dt$$

$$\Rightarrow \; R(z) = \begin{cases} -\frac{(\lambda_2-\varepsilon)}{p} b z^p - z\,\|\gamma\|_1 & \text{if } z \geq 0 \\ -\frac{(\lambda_2-\varepsilon)}{p} b|z|^p + |z|\,\|\gamma\|_1 & \text{if } z \leq 0. \end{cases}$$

So we infer that $R(z) \to -\infty$ as $|z| \to \infty$.
Next let $y \in Y$. Using the Poincare–Wirtinger inequality, we have

$$\begin{aligned} R(y) &= \frac{1}{p}\left\|y'\right\|_p^p - \frac{(\lambda_2-\varepsilon)}{p}\|y\|_p^p - \int_0^b \gamma(t)y(t)\,dt \\ &\geq \frac{1}{p}\left(1 - \frac{(\lambda_2-\varepsilon)}{p}\right)\left\|y'\right\|_p^p - \|\gamma\|_q \frac{1}{\lambda_2^{\frac{1}{p}}}\left\|y'\right\|_p \\ &\Rightarrow \; R(y) \to +\infty \text{ as } \|y\| \to \infty, \;\; y \in Y, \end{aligned}$$

(because $\left\|y'\right\|_p$ is an equivalent norm on Y).

Finally we will show that $R(\cdot)$ satisfies (PS)–condition. To this end let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$ such that $|R(x_n)| \leq M$ and $m(x_n) \to 0$ as $n \to \infty$. Let $x_n^* \in \partial R(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. Such an element exists since $\partial R(x_n)$ is nonempty, w–compact in $W^{1,p}_{per}(T)^*$ and the norm of $W^{1,p}_{per}(T)^*$ is weakly lower semicontinuous. Let $J_1, J_2, G : W^{1,p}_{per}(T) \to \mathbf{R}$ be defined by

$$J_1(x) = \frac{1}{p}\left\|x'\right\|_p^p, \;\; J_2(x) = \frac{\lambda_2-\varepsilon}{p}\|x\|_p^p \;\text{ and } G(x) = \int_0^b \gamma(t)x(t)\,dt.$$

We have $R(x) = J_1 - J_2(x) - G(x)$. Hence

$$\partial R(x) = \partial(J_1 - J_2 - G)(x) \subseteq \partial J_1(x) - \partial J_2(x) - \partial G(x)$$

for all $x \in W^{1,p}_{per}(T)$.
Let $A : W^{1,p}_{per}(T) \to W^{1,p}_{per}(T)^*$ be the nonlinear operator defined by

$$< A(x), y > = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt \;\text{ for all } x, y \in W^{1,p}(T).$$

Here by $< \cdot, \cdot >$ we denote the duality brackets for the pair $(W^{1,p}_{per}(T), W^{1,p}_{per}(T)^*)$. It is easy to verify that $A(\cdot)$ is monotone, demicontinuous and coercive (see for example Kourogenis–Papageorgiou [10]). Then $\partial J_1(x) = A(x)$. Also if $\widehat{J}_2 : L^p(T) \to \mathbf{R}$ is defined by $\widehat{J}_2(x) = \frac{1}{p}\|x\|_p^p$, then $J_2 = \widehat{J}_2|_{W^{1,p}_{per}(T)}$. Since $W^{1,p}_{per}(T)$ is embedded continuously and densely in $L^p(T)$, using Theorem 2.2 of Chang [3], we have that $\partial J_2(x) = (\lambda_2 - \varepsilon)|x(\cdot)|^{p-2}x(\cdot) \in L^q(T)$. So finally we have

$$x_n^* = A(x_n) - (\lambda_2 - \varepsilon)|x_n|^{p-2}x_n - \gamma.$$

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W_{per}^{1,p}(T)$ is either uniformly (in $t \in T$) bounded from above or below. Suppose not. Then we can find a subsequence, still denoted by $\{x_n\}_{n\geq 1}$ such that

$$L_n = \max_T x_n \to +\infty \quad \text{and} \quad \ell_n = \min_T x_n \to -\infty \quad \text{as} \quad n \to \infty.$$

Extending periodically $x_n(\cdot)$ on $[0, 2b]$, for all $n \geq 1$ large we can find $\alpha_n, \beta_n, \theta_n, \eta_n \in [0, 2b]$ such that

$$\begin{aligned} &x_n(\alpha_n) = x_n(\beta_n) = 0, \;\; x_n(t) > 0 \;\text{ for } t \in (\alpha_n, \beta_n), \\ &\max\,[x_n(t) : t \in [\alpha_n, \beta_n]] = L_n \end{aligned}$$

and

$$\begin{aligned} &x_n(\theta_n) = x_n(\eta_n) = 0, \;\; x_n(t) < 0 \;\text{ for } t \in (\theta_n, \eta_n), \\ &\min\,[x_n(t) : t \in [\theta_n, \eta_n]] = \ell_n \end{aligned}$$

We will show that there exists $\xi_1 > 0$ or $\xi_2 > 0$ such that

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_{\alpha_n}^{\beta_n} |x_n(t)|^p dt \geq \xi^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt, \tag{9}$$

and

$$\int_{\theta_n}^{\eta_n} |x_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_{\theta_n}^{\eta_n} |x_n(t)|^p dt \geq \xi^2 \int_{\theta_n}^{\eta_n} |x_n'(t)|^p dt. \tag{10}$$

Suppose not. Then at least for a subsequence, we will have

$$\begin{aligned} \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_{\alpha_n}^{\beta_n} |x_n(t)|^p dt &\leq \delta_n^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt \\ \int_{\theta_n}^{\eta_n} |x_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_{\theta_n}^{\eta_n} |x_n(t)|^p dt &\leq \delta_n^2 \int_{\theta_n}^{\eta_n} |x_n'(t)|^p dt \end{aligned}$$

with $\delta_n^1, \delta_n^2 \downarrow 0$ as $n \to \infty$. Also we may assume that $\alpha_n \to \alpha, \beta_n \to \beta, \theta_n \to \theta$ and $\eta_n \to \eta$ We will work with the first of the last two inequalities, since the arguments for the second are similar. Using Poincare's inequality on (α_n, β_n), we have

$$(\lambda_2 - \varepsilon) \int_{\alpha_n}^{\beta_n} |x_n(t)|^p dt \leq \frac{\lambda_2 - \varepsilon}{\overset{\circ}{\lambda}_1((\alpha_n, \beta_n))} \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt$$

Here by $\mathring{\lambda}_1\ ((\alpha_n, \beta_n)) > 0$ we denote the first (principal) eigenvalue of the one dimensional p–Laplacian on (α_n, β_n) with Dirichlet boundary conditions (i.e. of $(-\Delta_p, W_0^{1,p}(\alpha_n, \beta_n))$). From Otani [11] we know that

$$\mathring{\lambda}_1\ ((\alpha_n, \beta_n)) = \frac{1}{(\beta_n - \alpha_n)^p} \frac{p-1}{p^p} \beta\left(\frac{1}{p}, \frac{1}{q}\right)^p$$

where $\beta(x, y)$ is the beta function defined by $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$, $x, y > 0$. Note that $(\beta_n - \alpha_n) \leq |T|$ and so $\mathring{\lambda}_1\ (T) \leq \mathring{\lambda}_1\ ((\alpha_n, \beta_n))$. Also $\lambda_2 \leq \mathring{\lambda}_1\ (T)$. Then we have

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt - \frac{\lambda_2 - \varepsilon}{\mathring{\lambda}_1\ ((\alpha_n, \beta_n))} \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt \leq \delta_n^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt$$

Divide by $\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt$. We obtain

$$1 - \frac{\lambda_2 - \varepsilon}{\mathring{\lambda}_1\ ((\alpha_n, \beta_n))} \leq \delta_n^1$$

Passing to the limit as $n \to \infty$, we have

$$1 \leq \frac{\lambda_2 - \varepsilon}{\mathring{\lambda}_1\ (T)} < 1 \quad \left(\text{since } \mathring{\lambda}_1\ (T) \leq \mathring{\lambda}_1\ ((\alpha, \beta))\right)$$

a contradiction. Note that if $\alpha = 0, \beta = b$ then $\theta = \eta = b$ and vice versa. So we conclude that one of the inequalities (9) or (10) holds.

Suppose that (9) holds (the argument is similar if (10) holds). Define the functions $u_n : [0, 2b] \to \mathbf{R}$ as follows:

$$u_n(t) = \begin{cases} x_n(t) & \text{if } t \in [\alpha_n, \beta_n] \\ 0 & \text{if } t \in [0, 2b] \setminus [\alpha_n, \beta_n]. \end{cases}$$

Evidently $u_n \in W_0^{1,p}(0, 2b)$. Also let $w_n(t) = u_n(t)$ for all $t \in T$ if $\beta_n \leq b$, while if $\beta_n > b$ set $w_n(t) = u_n(t + b)$ for $t \in [0, \beta_n - b]$ and $w_n(t) = u_n(t)$ for $t \in [\beta_n - b, b]$. Clearly then $w_n \in W^{1,p}(T)$ and $\|w_n\|_{1,p,T} = \|u_n\|_{1,p,[0,2b]}$

(hence by $\|\cdot\|_{1,p,T}$ (resp. $\|\cdot\|_{1,p,[0,2b]}$) we denote the norm of the Sobolev space $W^{1,p}(T)$ (resp. of $W^{1,p}(0,2b)$)). We have

$$|<x_n^*, y>| \le \varepsilon_n \|y\| \quad \text{for all } y \in W^{1,p}_{per}(0,2b)$$

Let $y = w_n$. We obtain

$$\begin{aligned}
&\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_{\alpha_n}^{\beta_n} |x_n(t)|^p dt - \int_{\alpha_n}^{\beta_n} \gamma(t) x_n(t)\, dt \\
&\le \varepsilon_n \|u_n\|_{1,p,[0,2b]} \\
&\Rightarrow \xi_1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt - \int_{\alpha_n}^{\beta_n} \gamma(t) x_n(t)\, dt \le \varepsilon_n \|u_n\|_{1,p,[0,2b]} \\
&\Rightarrow \xi_1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p dt \le c_1 \|x_n\|_{1,p,(\alpha_n,\beta_n)}
\end{aligned}$$

for some $c_1 > 0$ (since $u_n|_{(\alpha_n,\beta_n)} = x_n$).

Using Poincare's inequality in the space $W_0^{1,p}(\alpha_n, \beta_n)$ we obtain that $\|x_n\|_{1,p,(\alpha_n,\beta_n)} \le c_2$ for some $c_2 > 0$ and all $n \ge 1$. So we have $\int_{\alpha_n}^{\beta_n} |x_n(t)|\, dt \le c_3$ for some $c_3 > 0$ and all $n \ge 1$. Therefore

$$\begin{aligned}
x_n(t) \le \int_{\alpha_n}^{t} |x_n'(s)|\, ds \le c_3 \quad &\text{for all } t \in [\alpha_n, \beta_n] \\
\Rightarrow \max\{x_n(t) : t \in [\alpha_n, \beta_n]\} = L_n \le c_3.&
\end{aligned}$$

However, by hypothesis $L_n \to +\infty$ as $n \to \infty$ and we have a contradiction. Thus we have proved that $\{x_n\}_{n\ge1}$ is uniformly (in $t \in T$) bounded from above or below. To fix things we will assume that it is bounded from above. So $x_n(t) \le c_4$ for all $t \in T$ and all $n \ge 1$. Recall that

$$|<x_n^*, y>| \le \varepsilon_n \|y\| \quad \text{for all } y \in W^{1,p}_{per}(T), \quad \text{with } \varepsilon_n \downarrow 0.$$

Let $y = 1$. We obtain

$$\begin{aligned}
&\left|(\lambda_2 - \varepsilon) \int_0^b |x_n(t)|^{p-2} x_n(t)\, dt\right| \le c_5 \quad \text{for some } c_5 > 0 \text{ and } \forall n \ge 1 \\
&\Rightarrow \left|\int_{\{x_n \ge 0\}} |x_n(t)|^{p-2} x_n(t)\, dt + \int_{\{x_n \le 0\}} |x_n(t)|^{p-2} x_n(t)\, dt\right| \\
&\le c_6 \quad \left(c_6 = \frac{1}{\lambda_2 - \varepsilon} c_5\right).
\end{aligned}$$

Since $x_n(t) \leq c_4$ for all $t \in T$ and all $n \geq 1$, we see that

$$\begin{aligned}
&\left| \int_{\{x_n \leq 0\}} |x_n(t)|^{p-2} x_n(t)\, dt \right| \leq c_7 \quad \left(c_7 = c_6 + c_4^{p-1} b\right) \\
&\Rightarrow \int_{\{x_n \leq 0\}} |x_n(t)|^{p-2} (-x_n(t))\, dt \leq c_7 \\
&\Rightarrow \sup_{n \geq 1} \int_{\{x_n \leq 0\}} |x_n(t)|^{p-1}\, dt \leq c_7 \\
&\Rightarrow \sup_{n \geq 1} \int_0^b |x_n(t)|^{p-1}\, dt \leq c_8 \quad (c_8 = c_6 + 2c_4^{p-1} b);
\end{aligned}$$

i.e. $\sup_{n \geq 1} \|x_n\|_p^{p-1} \leq c_8$.

Now let $x_n = z_n + y_n$, $z_n \in Z = \mathbf{R}$ and $y_n \in Y$, $n \geq 1$. We have

$$\begin{aligned}
&< x_n^*, y_n > \leq \varepsilon_n \|y_n\| \\
&\Rightarrow \int_0^b |y_n'(t)|^p dt - (\lambda_2 - \varepsilon) \int_0^b |x_n(t)|^{p-1} y_n(t)\, dt \\
&\qquad - \int_0^b \gamma(t) y_n(t)\, dt \leq c_1 \|y_n\|. \qquad (11)
\end{aligned}$$

From the Poincare–Wirtinger inequality we know that

$$\|y_n\|_p^p \leq \frac{1}{\lambda_2} \left\|y_n'\right\|_p^p.$$

Using this fact in (11) we obtain

$$\begin{aligned}
\left\|y_n'\right\|_p^p &\leq (\lambda_2 - \varepsilon) \|x_n\|_p^{p-1} \|y_n\|_p + \|\gamma\|_q \|y_n\|_p \\
&\quad + c_9 \left\|y_n'\right\|_p \quad \text{for some} c_9 > 0 \\
&\leq \left[(\lambda_2 - \varepsilon) c_8 \frac{1}{\lambda_2^{1/p}} + \|\gamma\|_q \frac{1}{\lambda_2^{1/q}} + c_9 \right] \left\|y_n'\right\|_p \\
&\Rightarrow \{y_n' = x_n'\}_{n \geq 1} \subseteq L^p(T) \ \text{ is bounded.} \qquad (12)
\end{aligned}$$

Finally recall that $|R(x_n)| \leq M$ for all $n \geq 1$. So using the boundedness of $\{x_n'\}_{n \geq 1} \subseteq L^p(T)$ we have

$$\begin{aligned}
&-\frac{1}{p} \left\|x_n'\right\|_p^p + \frac{\lambda_2 - \varepsilon}{p} \|x_n\|_p^p \leq M + \|\gamma\|_q \|x_n\|_p \\
&\Rightarrow \|x_n\|_p^p \leq c_{10} (1 + \|\gamma\|_q \|x_n\|_p) \quad \text{for some } c_{10} > 0 \\
&\Rightarrow \{x_n\}_{n \geq 1} \subseteq L^p(T) \ \text{ is bounded.}
\end{aligned}$$

So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T)$ and $x_n \to x$ in $L^p(T)$ as $n \to \infty$. We have

$$\begin{aligned} < x_n^*, x_n - x > &= < A(x_n), x_n - x > \\ &\quad -(\lambda_2 - \varepsilon) \int_0^b |x_n(t)|^{p-2} x_n(t)(x_n(t) - x(t))\, dt \\ &\quad - \int_0^b \gamma(t)(x_n(t) - x(t))\, dt \\ &\leq \varepsilon_n \|x_n - x\| \end{aligned}$$

But
$\int_0^b |x_n(t)|^{p-2} x_n(t)(x_n(t) - x(t))\, dt \to 0$ and $\int_0^b \gamma(t)(x_n(t) - x(t))\, dt \to 0$ as $n \to \infty$. Thus we obtain that

$$\overline{\lim} < A(x_n), x_n - x > \leq 0.$$

Since $A(\cdot)$ is maximal monotone (being monotone, demicontinuous, every where defined, see Zeidler [14]) it has the generalized pseudomonotone property (see Browder–Hess [2]) and so

$$< A(x_n), x_n > \to < A(x), x > \quad \text{as } n \to \infty \Rightarrow \left\| x_n^{'} \right\|_p \to \left\| x^{'} \right\|_p \quad \text{as } n \to \infty.$$

Because $x_n^{'} \xrightarrow{w} x^{'}$ in $L^p(T)$ and the latter space is uniformly convex (thus has the Kadec–Klee property), we conclude that $x_n \to x$ in $W^{1,p}_{per}(T)$ as $n \to \infty$. Therefore $R(\cdot)$ satisfies the (PS)–condition.

Apply Theorem 1 to obtain $\phi \in W^{1,p}_{per}(T)$, $\phi \neq 0$ such that $0 \in \partial R(\phi) = A(\phi) - (\lambda_2 - \varepsilon)|\phi|^{p-2}\phi - \gamma$. Then for all $\xi \in C_0^\infty(0, b)$ we have

$$\begin{aligned} (\lambda_2 - \varepsilon) \int_0^b |\phi(t)|^{p-2}\phi(t)\xi(t)\, dt + \int_0^b \gamma(t)\xi(t)\, dt &= < A\phi), \xi > \\ &= \int_0^b |\phi^{'}(t)|^{p-2}\phi^{'}(t)\xi^{'}(t)\, dt \qquad (13) \end{aligned}$$

Since $|\phi(\cdot)|^{p-2}\phi(\cdot) + \gamma(\cdot) \in L^q(T)$, from the definition of the distributional derivative, it follows that $|\phi^{'}(\cdot)|^{p-2}\phi^{'}(\cdot) \in W^{1,q}(T)$ and

$$\left\{ \begin{array}{l} -(|\phi^{'}(t)|^{p-2}\phi^{'}(t))^{'} = (\lambda_2 - \varepsilon)|\phi(t)|^{p-2}\phi(t) + \gamma(t) \ \text{ a.e. on } T \\ \phi(0) = \phi(b) \end{array} \right\} \qquad (14)$$

Also from Green's formula (integration by parts), we have

$$\begin{aligned}\int_0^b (|\phi'(t)|^{p-2}\phi'(t))' y(t)\,dt &= |\phi'(b)|^{p-2}\phi'(b)y(b) - |\phi'(0)|^{p-2}\phi'(0)y(0)\\ &\quad -\int_0^b |\phi'(t)|^{p-2}\phi'(t)y'(t)\,dt\end{aligned}$$

for all $y \in W^{1,p}_{per}(T)$.

Using (13) and (14) we obtain

$$|\phi'(0)|^{p-2}\phi'(0)y(0) = |\phi'(b)|^{p-2}\phi'(b)y(b).$$

Let $y \in W^{1,p}_{per}(T)$ be such that $y(0) = y(b) = 1$. We have

$$|\phi'(0)|^{p-2}\phi'(0) = |\phi'(b)|^{p-2}\phi'(b).$$

But the map $r \rightarrow \frac{1}{p}|r|^p$ is strictly convex and differentiable on $\mathbf{R}$. So its derivative $r \rightarrow |r|^{p-2}r$ is strictly monotone. Hence $\phi'(0) = \phi'(b)$, i.e. $\phi \in C^1(T)$ is a nontrivial solution of (7) and from the beginning of the proof we know that $\phi \geq 0$.

In a similar way we can prove the next proposition.

Proposition 3. Problem (7) has a solution $\psi \leq 0$, $\psi \neq 0$.

4 Main Result

We start this section by introducing the notions of upper and lower solutions for problem (2).

Definition.

(a) A function $u \in C^1(T)$ is said to be an "upper solution" for problem (2) if $|u'|^{p-2}u' \in W^{1,q}(T)$ and $-(|u'(t)|^{p-2}u'(t))' \geq f_1(t, u(t))$ a.e. on T, $u(0) = u(b)$, $u'(0) \leq u'(b)$.

(b) A function $v \in C^1(T)$ is said to be an "lower solution" for problem (2) if $|v'|^{p-2}v' \in W^{1,q}(T)$ and $-(|v'(t)|^{p-2}v'(t))' \leq f_0(t, v(t))$ a.e. on T, $v(0) = v(b)$, $v'(0) \geq v'(b)$.

Evidently the functions ϕ and ψ obtained in Propositions 2 and 3 are upper and lower solutions respectively for problem (2).

Now let $\tau : T\times\mathbf{R} \to \mathbf{R}$ and $\beta : T\times\mathbf{R} \to \mathbf{R}$ be respectively the truncation and penalty functions defined for all $(t,x) \in T \times \mathbf{R} \to \mathbf{R}$ by

$$\tau = \begin{cases} \phi(t) & \text{if } \phi(t) \leq x \\ x & \text{if } \psi(t) \leq x \leq \phi(t) \\ \psi(t) & \text{if } x \leq \psi(t) \end{cases} \quad \text{and} \quad \beta = \begin{cases} 1 & \text{if } \phi(t) < x \\ 0 & \text{if } \psi(t) \leq x \leq \phi(t) \\ -1 & \text{if } x < \psi(t). \end{cases}$$

Because $\phi \geq 0$, $\phi \neq 0$ and by virtue of hypothesis $H(f)(iv)$ we see that $f(t,\phi(t)) \geq 0$ a.e. on T and the inequality is strict on a set of positive Lebesque measure. So $\int_0^b f(t,\phi(t))\,dt > 0$. Similarly we obtain that $\int_0^b f(t,\psi(t))\,dt < 0$. Thus we can find $\lambda > 0$ small enough so that $\int_0^b f(t,\phi(t))\,dt > \lambda b$ and $\int_0^b f(t,\psi(t))\,dt < -\lambda b$. With $\lambda > 0$ chosen this way, we define $h(t,x) = g(t,x) - \lambda\beta(t,x)$ with $g(t,x) = f(t,\tau(t,x))$. Clearly $h(\cdot,\cdot 0$ is Borel measurable and for almost all $t \in T$ and all $x \in \mathbf{R}$, we have $|h(t,x)| \leq \gamma(t)$ with $\gamma \in L^q(T)$ (for example we can take $\gamma = \gamma_M + \lambda$ with $M = \max\{\|\phi\|_\infty, \|\psi\|_\infty\}$).

Let $J, H : W^{1,p}_{per}(T) \to \mathbf{R}$ be the fuctionals defined by

$$J(x) = \frac{1}{p}\left\|x'\right\|_p^p \quad \text{and} \quad H(x) = \int_0^b \int_0^{x(t)} h(t,r)\,dr\,dt.$$

Note that $J \in C^1(W^{1,p}_{per}(T))$ with $\partial T(x) = J'(x) = A(x)$ (see the proof of Proposition 2) and H is locally Lipschitz (see Chang [3]). So if we set $R(x) = J(x) - H(x)$, then $R : W^{1,p}_{per}(T) \to \mathbf{R}$ is locally Lipschitz. In the next proposition we establish the existence of nontrivial critical points for the functional $R(\cdot)$.

Proposition 4. If hypotheses $H(f)$ hold, then $R(\cdot)$ has a nontrivial critical point $x(\cdot)$.

Proof. Again consider the decomposition $W^{1,p}_{per}(T) = Z \oplus Y$ with $Z = \mathbf{R}$ and $Y = \{y \in W^{1,p}_{per}(T) : \int_0^b y(t)\,dt = 0\}$. For $z \in Z = \mathbf{R}$ we have

$$R(z) = -\int_0^b \int_0^z h(t,r)\,dr\,dt = -\int_0^b \int_0^z g(t,r)\,dr\,dt + \lambda \int_0^b \int_0^z \beta(t,r)\,dr\,dt.$$

First suppose that $z \geq \|\phi\|_\infty > 0$. From the definitions of $g(t,r)$ and $\beta(t,r)$ we have

$$\begin{aligned} R(z) &= -\int_0^b \int_0^{\phi(t)} f(t,r)\,dr\,dt - \int_0^b \int_{\phi(t)}^z f(t,\phi(t))\,dt + \lambda \int_0^b \int_{\phi(t)}^z dr\,dt \\ &= -\int_0^b F(t,\phi(t))\,dt - \int_0^b (f(t,\phi(t)) - \lambda)(z - \phi(t))\,dt \end{aligned}$$

Note that $\left|\int_0^b F(t,\phi(t))\,dt\right| \leq \|\gamma\|_1 \|\phi\|_\infty$ and $\left|\int_0^b (f(t,\phi(t)) - \lambda)\phi(t)\,dt\right| \leq (\|\gamma\|_1 + \lambda b)\,\|\phi\|_\infty$.

Also $\int_0^b (f(t,\phi(t)) - \lambda)z\,dt = z\left(\int_0^b f(t,\phi(t))\,dt - \lambda b\right) > 0$ (recall the choice of $\lambda > 0$). Thus we have

$$R(z) \to -\infty \quad \text{as} \quad z \to +\infty.$$

If $z \leq -\|\psi\|_\infty < 0$, we have

$$\begin{aligned} R(z) &= -\int_0^b \int_0^{\psi(t)} f(t,r)\,dr\,dt - \int_0^b \int_{\psi(t)}^z f(t,\psi(t))\,dr\,dt \\ &\quad -\lambda \int_0^b \int_{\psi(t)}^z dr\,dt \\ &= -\int_0^b F(t,\psi(t))\,dt - \int_0^b (f(t,\psi(t)) + \lambda)(z - \psi(t))\,dt. \end{aligned}$$

Again $\left|\int_0^b F(t,\psi(t))\,dt\right| \leq \|\gamma\|_1 \|\psi\|_\infty$ and $\left|\int_0^b (f(t,\psi(t)) + \lambda)\psi(t)\,dt\right| \leq (\|\gamma\|_1 + \lambda b)\,\|\psi\|_\infty$. Also our choice of $\lambda > 0$ implies that $-\int_0^b (f(t,\psi(t)) + \lambda)z\,dt = -z\left(\int_0^b f(t,\psi(t))\,dt + \lambda b\right) < 0$. Thus we have

$$R(z) \to +\infty \quad \text{as} \quad z \to -\infty.$$

Therefore we can say that $R(z) \to -\infty$ as $|z| \to \infty$.
Next let $y \in Y$. We have

$$R(y) = \frac{1}{p}\left\|y'\right\|_p^p - \int_0^b \int_0^{y(t)} h(t,r)\,dr\,dt \geq \frac{1}{p}\left\|y'\right\|_p^p - \|\gamma\|_1 \|y\|_\infty .$$

Since $W^{1,p}_{per}(T)$ is embedded continuously in $C(T)$ and by using the Poincare–Wirtinger inequality, we obtain

$$\begin{aligned} R(y) &\geq \frac{1}{p}\left\|y'\right\|_p^p - c_1\left\|y'\right\|_p \quad \text{for some} c_1 > 0 \\ \Rightarrow \quad & R(y) \to +\infty \ \text{ as } \|y\| \to \infty,\ y \in Y. \end{aligned}$$

Finally let us check that $R(\cdot)$ satisfies the (PS)–condition. To this end let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$ such that $|R(x_n)| \leq M$ and $m(x_n) \to 0$ as $n \geq \infty$. Let $x_n^* \in \partial R(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We know that $\partial R(x_n) \subseteq \partial J(x_n) - \partial H(x_n)$ (see Section 2) and $\partial J(x_n) = A(x_n)$. Hence $x_n^* = A(x_n) - \theta_n$ with $\theta_n \in \partial H(x_n)$. Let $\widehat{H} : L^p(T) \to \mathbf{R}$ be defined by $\widehat{H}(x) = \int_0^b \int_0^{x(t)} h(t,r)\,dr\,dt$. Evidently $H = \widehat{H}|_{W^{1,p}_{per}(T)}$. Using Theorem 2.2 of Chang [3], we have $\partial H(x_n) \subseteq \partial \widehat{H}(x_n) \subseteq L^q(T)$. Moreover, by definition (see Section 2)

$$\partial \widehat{H}(x_n) = \{\theta \in L^q(T) : \int_0^b \theta(t)y(t)\,dt \leq \widehat{H}^0(x_n;y) \ \text{ for all } y \in L^p(T)\},$$

where

$$\begin{aligned} \widehat{H}^0(x_n;y) &= \varlimsup_{\substack{a\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}\left[\widehat{H}(x_n + a + \lambda y) - \widehat{H}(x_n + a)\right] \\ &= \varlimsup_{\substack{a\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}\int_0^b \int_{(x_n+a)(t)}^{(x_n+a+\lambda y)(t)} h(t,r)\,dr\,dt. \end{aligned}$$

Performing a change of variables according to
$r(\eta) = x_n(t) + a(t) + \eta\lambda y(t)$ and using Fatou's lemma, we obtain

$$\begin{aligned} \widehat{H}^0(x_n;y) \ &\leq \ \int_0^b \varlimsup_{\substack{a\to 0\\ \lambda\downarrow 0}} \int_0^1 h(t, x_n(t) + a(t) + \eta\lambda y(t))\,d\eta\,dt \\ &\leq \ \int_{\{y>0\}} h_1(t,x_n(t))y(t)\,dt + \int_{\{y<0\}} h_0(t,x_n(t))y(t)\,dt \\ &\Rightarrow \ \int_0^b \theta_n(t)y(t)\,dt \leq \int_{\{y>0\}} h_1(t,x_n(t))y(t)\,dt \\ &\qquad + \int_{\{y<0\}} h_0(t,x_n(t))y(t)\,dt \ \text{ for all } y \in L^p(T) \\ &\Rightarrow \ h_0(t,x_n(t)) \leq \theta_n(t) \leq h_1(t,x_n(t)) \ \text{ a.e. on } T \\ &\Rightarrow \ \{\theta_n\}_{n\geq 1} \subseteq L^q(T) \ \text{is bounded.} \end{aligned}$$

Let $x_n = z_n + y_n$ with $z_n \in Z = \mathbf{R}$ and $y_n \in Y$, $n \geq 1$. We have

$$\begin{aligned} &|< x_n^*, y_n >| = \left| \left\| y_n' \right\|_p^p - \int_0^b \theta_n(t) y_n(t)\, dt \right| \leq \varepsilon_n \left\| y_n \right\| \quad (\varepsilon_n \downarrow 0) \\ \Rightarrow\ & \left\| y_n' \right\|_p^p \leq (\varepsilon_n + \left\| \theta \right\|_q) c_2 \left\| y_n' \right\|_p \quad \text{for some } c_2 > 0 \end{aligned}$$

(here we have used the Poincare–Wirtinger inequality). Therefore $\{y_n' = x_n'\}_{n \geq 1} \subseteq L^p(T)$ is bounded. We claim that $\{x_n\}_{n \geq 1} \subseteq C(T)$ is bounded. Suppose not. Then because $\{x_n'\}_{n \geq 1} \subseteq L^p(T)$ is bounded, we must have $L_n = \max_T x_n \to -\infty$ or $\ell_n = \min_T x_n \to +\infty$ as $n \to \infty$. Suppose the first is true (the analysis being similar if the second holds). We have

$$\begin{aligned} & |R(x_n)| \leq M \quad \text{for all } n \geq 1 \\ \Rightarrow\ & -\frac{1}{p} \left\| x_n' \right\|_p^p + \int_0^b \int_0^{x_n(t)} h(t,r)\, dr\, dt \leq M \\ \Rightarrow\ & -\frac{1}{p} \left\| x_n' \right\|_p^p + \int_0^b \int_0^{L_n} h(t,r)\, dr\, dt - \int_0^b \int_{x_n(t)}^{L_n} h(t,r)\, dr\, dt \leq M \end{aligned}$$

Note that

$$\begin{aligned} & \int_0^b \int_{x_n(t)}^{L_n} h(t,r)\, dr\, dt \leq \int_0^b \gamma(t)(L_n - x_n(t))\, dt \leq \left\| \gamma \right\|_1 \left\| L_n - x_n \right\|_\infty \\ \Rightarrow\ & -\int_0^b \int_{x_n(t)}^{L_n} h(t,r)\, dr\, dt \geq -\left\| \gamma \right\|_1 \left\| L_n - x_n \right\|_\infty . \end{aligned}$$

Choose $t_n \in T$ such that $L_n = x_n(t_n)$, $n \geq 1$. We have

$$\begin{aligned} & x_n(t_n) - x_n(t) = \int_t^{t_n} x_n'(s)\, ds, \quad t \in T \\ \Rightarrow\ & \left\| x_n(t_n) - x_n(t) \right\| = \left\| L_n - x_n(t) \right\| \leq \sup_{n \geq 1} \left\| x_n' \right\|_1 \leq c_3 \\ & \text{for some } c_3 > 0 \text{ and all } t \in T \\ \Rightarrow\ & \left\| L_n - x_n \right\|_\infty \leq c_3 \end{aligned}$$

Thus it follows that for $n \geq 1$ large enough (so that $L_n \leq -\left\| \psi \right\|_\infty$), we have

$$\int_0^b \int_0^{L_n} h(t,r)\, dr\, dt \ \leq\ M + \sup_{n \geq 1} \frac{1}{p} \left\| x_n' \right\|_p^p + \left\| \gamma \right\|_1 c_3 = c_4 < +\infty$$

$$\begin{aligned}\Rightarrow \int_0^b \int_0^{L_n} h(t,r)\,dr\,dt &= \int_0^b \int_0^{\psi(t)} g(t,r)\,dr\,dt + \int_0^b \int_{\psi(t)}^{L_n} g(t,r)\,dr\,dt \\ &\quad + \lambda \int_0^b \int_{\psi(t)}^{L_n} dr\,dt \\ &= \int_0^b F(t,\psi(t))\,dt + \int_0^b f(t,\psi(t))(L_n - \psi(t))\,dt \\ &\quad + \lambda \int_0^b (L_n - \psi(t))\,dt \le c_4. \qquad (15)\end{aligned}$$

Note that from the choice of $\lambda > 0$, we have $\int_0^b (f(t,\psi(t)) + \lambda)\,dt < 0$. Also $\left|\int_0^b F(t,\psi(t))\,dt\right|$, $\left|\int_0^b f(t,\psi(t))\psi(t)\,dt\right| \le c_5$ for some $c_5 > 0$. since $L_n \to -\infty$, we have $L_n \int_0^b (f(t,\psi(t)) + \lambda)\,dt \to +\infty$ as $n \to \infty$ and we have a contradiction to (14). So indeed $\{x_n\}_{n\ge1} \subseteq C(T)$ is bounded and this together with the boundedness of $\{x_n' = y_n'\}_{n\ge1} \subseteq L^p(T)$ implies that $\{x_n\}_{n\ge1} \subseteq W^{1,p}(T)$ is bounded. Arguing as in the proof of Proposition 2, we can extract a strongly convergent subsequence of $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{per}(T)$. Thus $R(\cdot)$ satisfies the (PS)–condition. Apply Theorem 1 to obtain $x \in W^{1,p}_{per}(T)$, $x \ne 0$ such that $0 \in \partial R(x)$.

Using this proposition, we can have an existence theorem for problem (2).

Theorem 5. If hypotheses $H(f)$ hold, then problem (2) has a nontrivial solution $x(\cdot)$.

Proof. Let $x \in W^{1,p}_{per}(T)$ be a nontrivial critical point of the functional $R(\cdot)$. It exists by Proposition 4. We have $0 \in \partial R(x) \subseteq A(x) - \partial H(x)$, hence $A(x) = \theta$ for $\theta \in \partial H(x)$. Arguing as at the end of the proof of Proposition 2, we can have that

$$\left\{\begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' = \theta(t) \ \text{ a.e. on } T \\ x(0) = x(b), \ \ x'(0) = x'(b). \end{array}\right\}$$

Also let $V, B : W^{1,p}_{per}(T) \to \mathbf{R}$ be the locally Lipschitz functionals defined by $V(x) = \int_0^b \int_0^{x(t)} g(t,r)\,dr\,dt$ and $B(x) = \int_0^b \int_0^{x(t)} \beta(t,r)\,dr\,dt$ (see Chang [?]). We see that $H(x) = V(x) - \lambda B(x)$. Hence we have that

$\theta \in \partial H(x) \subseteq \partial V(x) - \lambda \partial B(x)$ (see Section 2) and so $\theta = v - \lambda w$ with $v \in \partial V(x)$ and $w \in \partial B(x)$. Since ψ is a lower solution of problem (2), we have

$$\int_0^b \left(|x'(t)|^{p-2}x'(t)y'(t) - |\psi'(t)|^{p-2}\psi'(t)y'(t)\right)\, dt \geq$$
$$\int_0^b \left(v(t) - \lambda w(t) - f_0(t, \psi(t))\right) y(t)\, dt \tag{16}$$

for all $y \in W^{1,p}_{per}(T) \cap L^p(T)_+$.
Let $y = (\psi - x)_+ \in W^{1,p}_{per}(T) \cap L^p(T)_+$ and recall that

$$(\psi - x)_+'(t) = \begin{cases} (\psi - x)'(t) & \text{if } (\psi - x)(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

(see Gilbarg–Trudinger [8]). We have

$$\begin{aligned}
&\int_0^b \left(|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t)\right)(\psi - x)_+'(t)\, dt \\
= \; & \int_{\{\psi > x\}} \left(|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t)\right)(\psi'(t) - x'(t))\, dt \\
\leq \; & -2^{2-p} \int_{\{\psi > x\}} |\psi'(t) - x'(t)|^p\, dt \\
= \; & -2^{2-p} \int_0^b |(\psi - x)_+'(t)|^p dt \leq 0.
\end{aligned} \tag{17}$$

Also we have that

$$\int_0^b \left(v(t) - f_0(t, \psi(t))\right)(\psi - x)_+(t)\, dt =$$
$$\int_{\{\psi > x\}} \left(v(t) - f_0(t, \psi(t))\right)(\psi - x)(t)\, dt$$

Recall that since $v \in \partial H(x)$, $v(t) \geq g_0(t, x(t))$ a.e. on T. Also from the definition of $g(t, r)$ we see that for almost all $t \in \{x < \psi\}$, we have $f_0(t, \psi(t)) = g_0(t, x(t))$. So $f_0(t, \psi(t)) \leq v(t)$ a.e. on $\{x < \psi\}$. Hence we have

$$\int_0^b \left(v(t) - f_0(t, \psi(t))\right)(\psi - x)_+(t)\, dt \geq 0. \tag{18}$$

Finally note that since $w \in \partial B(x)$, we have
$\beta_0(t, x(t)) \leq w(t) \leq \beta_1(t, x(t))$ a.e. on T. But from the definition of the

penalty function $\beta(t,r)$, we see that for almost all $t \in \{x < \psi\}$ we have $\beta_0(t,x(t)) = \beta_1(t,x(t)) = -1$, hence $w(t) = -1$ a.e. on $\{x < \psi\}$. Thus we have

$$\begin{aligned} -\lambda \int_0^b w(t)(\psi - x)_+(t)\,dt &= -\lambda \int_{\{\psi > x\}} (-1)(\psi - x)(t)\,dt \\ &= \lambda \int_0^b (\psi - x)_+(t)\,dt \qquad (19) \end{aligned}$$

Using (17), (18) and (19) in (16), we have

$$\lambda \int_0^b (\psi - x)_+(t)\,dt = 0\ , \quad \lambda > 0, \Rightarrow \psi(t) \le x(t) \quad \text{for all } t \in T.$$

Similarly we obtain that $(x(t) \le \phi(t)$ for all $t \in T$. Therefore we have that for all $t \in T$ $\psi(t) \le x(t) \le \phi(t)$. Recall that $-(|x'(t)|^{p-2}x'(t))' = \theta(t)$ a.e. on T with $h_0(t,x(t)) \le \theta(t) \le h_1(t,x(t))$ a.e. on T. Using the definitions of h, h_0 and h_1 and the fact that $\psi(t) \le x(t) \le \phi(t)$ for all $t \in T$, we can check that $h_0(t,x(t)) = f_0(t,x(t))$ and $h_1(t,x(t)) = f_1(t,x(t))$ a.e. on T. Thus we conclude that

$$\left\{ \begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' \in [f_0(t,x(t)), f_1(t,x(t))] \quad \text{a.e. on } T \\ x(0) = x(b), \quad x'(0) = x'(b). \end{array} \right\}$$

$\Rightarrow \quad x(\cdot)$ is a nontrivial solution of problem (2).

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Global Stability of Lotka-Volterra Prey-Predator Systems with Spatial Heterogeneity *

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(Submitted by: H.I. Freedman)

Abstract

We study a 2-patch Lotka-Volterra type prey-predator model. We analyze the existence condition on a positive equilibrium point (coexistence state) and its global stability, where we allow the migration only for prey or predators among heterogeneous patches. It is shown that spatial heterogeneity and migration of prey or predators change a neutrally stable positive equilibrium point to a globally asymptotically stable one.

1 Introduction

Population's persistence in patchy environments is expected as the result from an interaction between within patch local dynamics and the

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dispersal of the species among spatial heterogeneous patches. Murdoch *et al.* [1] extended a pure (without intraspecific competition among prey) Lotka-Volterra (L-V) type prey and predator model to a 2-patch model. They showed by numerical simulations that the migration of prey (or predator) brings the model to have asymptotically stable dynamics. Kawasaki and Teramoto [2] analyzed a pure L-V type prey and predator model with N-patches which are linked by predator's migration. They proved by using a Lyapunov function that the spatial heterogeneity among patches and migration of predator derive a globally asymptotically stable dynamics from a neutrally stable one. Namba *et al.* [3] considered the L-V predator migration model with intraspecific competition among prey and showed that productivity in patches and patterns of connectivity between patches differently influence persistence of the model.

In this paper, we analyze a 2-patch L-V type prey and predator model which was proposed by Godfray and Pacala [4] and described by the following system of differential equations:

$$\begin{aligned}
\dot{H}_1 &= a_1H_1 - b_1H_1P_1 - e_HH_1 + z_1e_H(H_1+H_2),\\
\dot{P}_1 &= fb_1H_1P_1 - d_1P_1 - e_PP_1 + q_1e_P(P_1+P_2),\\
\dot{H}_2 &= a_2H_2 - b_2H_2P_2 - e_HH_2 + z_2e_H(H_1+H_2),\\
\dot{P}_2 &= fb_2H_2P_2 - d_2P_2 - e_PP_2 + q_2e_P(P_1+P_2).
\end{aligned} \tag{1}$$

Here H_i and P_i are the densities of prey and predators in patch i ($i = 1, 2$). The a_i is the prey growth rate in the absence of predation, b_i is the per-head attack rate of the predator, f is the efficiency of converting prey to predators and d_i is the death rate of the predator. The instantaneous rates at which prey and predators leave a patch are e_H and e_P. The z_i and q_i are the probabilities for the prey and predator to immigrate to patch i ($i = 1, 2$) (i.e. $0 < z_i < 1,\ 0 < q_i < 1$) and satisfy

$$z_1 + z_2 = 1 \text{ and } q_1 + q_2 = 1,$$

so all emigrants are immigrants, that is, there is no mortality in transit. Further we assume $a_i, b_i, d_i, f, z_i, q_i > 0$ ($i = 1, 2$) and $0 \le e_H \le 1,\ 0 \le e_P \le 1$.

In this article, the dispersal of population is expressed by the differences of the local (within patch) population density. The spatial heterogeneity is expressed by the differences of parameter values in each patch, such as a growth rate, death rate and predation rate etc. We analyze system (1) by preventing either the predators or prey dispersion (i.e. $e_P = 0$ (Section 2) or $e_H = 0$ (Section 3)). The final section considers the effects of species migration among the heterogeneous patches on the coexistence of all species in the both patches. System (1) with intraspecific competition among the prey will be considered in the separate paper [5].

2 Model with Prey Migration

If there exists no migration of the both prey and predator (i.e. e_H, $e_P = 0$), the isolated patches dynamics are represented by the following pure L-V predator-prey equations, for $i = 1, 2$,

$$\begin{aligned} \dot{H}_i &= (a_i - b_i P_i) H_i, \\ \dot{P}_i &= (f b_i H_i - d_i) P_i. \end{aligned} \tag{2}$$

It is well known that system (2) has a neutrally stable non-trivial equilibrium point. We regard this as a reference system and consider the role of additional features of the model (e.g., prey or predators migration), that is, we analyze the problem whether the migration plays a "stabilizing" role or a "destabilizing" role on the reference system.

Note that the case where the parameter values of two patches are identical and prey (or predator) migration is symmetry, that is, $z_i = 1/2$ (or $q_i = 1/2$), system (1) again has a neutrally stable non-trivial equilibrium point. Hence two identical patches can be regarded as one larger isolated patch.

We first prevent predators movement ($e_P = 0$), so system (1) becomes as follows:

$$\begin{aligned} \dot{H}_1 &= a_1 H_1 - b_1 H_1 P_1 - e_H H_1 + z_1 e_H (H_1 + H_2), \\ \dot{P}_1 &= f b_1 H_1 P_1 - d_1 P_1, \\ \dot{H}_2 &= a_2 H_2 - b_2 H_2 P_2 - e_H H_2 + z_2 e_H (H_1 + H_2), \\ \dot{P}_2 &= f b_2 H_2 P_2 - d_2 P_2. \end{aligned} \tag{3}$$

There exist at most four nonnegative equilibrium points for (3):

$$\begin{aligned} E_{0000} &= (0,\ 0,\ 0,\ 0) \quad , & E_{+0++} &= (\bar{H}_1,\ 0,\ \bar{H}_2,\ \bar{P}_2), \\ E_{++++} &= (H_1^*,\ P_1^*,\ H_2^*,\ P_2^*) \quad , & E_{+++0} &= (\tilde{H}_1,\ \tilde{P}_1,\ \tilde{H}_2,\ 0), \end{aligned}$$

where $\bar{H}_1,\ \bar{H}_2,\ \bar{P}_2,\ H_1^*,\ P_1^*,\ H_2^*,\ P_2^*,\ \tilde{H}_1,\ \tilde{P}_1$ and $\tilde{H}_2$ are given by

$$\left\{ \begin{aligned} \bar{H}_1 &= e_H z_1 \bar{H}_2/(e_H z_2 - a_1), \\ \bar{H}_2 &= d_2/(f b_2), \\ \bar{P}_2 &= a_2/b_2 + e_H(z_2 \bar{H}_1 - z_1 \bar{H}_2)/(b_2 \bar{H}_2), \end{aligned} \right.$$

$$\left\{ \begin{aligned} H_1^* &= d_1/(f b_1), \\ P_1^* &= a_1/b_1 + e_H(z_1 H_2^* - z_2 H_1^*)/(b_1 H_1^*), \\ H_2^* &= d_2/(f b_2), \\ P_2^* &= a_2/b_2 + e_H(z_2 H_1^* - z_1 H_2^*)/(b_2 H_2^*), \end{aligned} \right.$$

$$\left\{ \begin{aligned} \tilde{H}_1 &= d_1/(f b_1), \\ \tilde{P}_1 &= a_1/b_1 + e_H(z_1 \tilde{H}_2 - z_2 \tilde{H}_1)/(b_1 \tilde{H}_1), \\ \tilde{H}_2 &= e_H z_2 \tilde{H}_1/(e_H z_1 - a_2). \end{aligned} \right.$$

It is easy to check that no equilibrium point E_{+0+0} exists. We are interested in a strictly positive equilibrium point (coexistence state) E_{++++}. It exists if and only if $P_i^* > 0$ $(i = 1, 2)$. So we obtain the following condition on the existence of a positive equilibrium point:

$$\frac{H_1^*(e_H - a_1)}{(H_1^* + H_2^*)e_H} < z_1 < \frac{H_1^* e_H + a_2 H_2^*}{(H_1^* + H_2^*)e_H}. \tag{4}$$

The Fig.1 illustrates this condition. Here, we define the regions as follows:

$$\begin{aligned} J_1 &= \left\{ (z_1,\ e_H) \mid 0 < z_1 < \frac{H_1^*(e_H - a_1)}{(H_1^* + H_2^*)e_H},\ 0 < e_H \leq 1 \right\}, \\ J_2 &= \left\{ (z_1,\ e_H) \mid \frac{H_1^*(e_H - a_1)}{(H_1^* + H_2^*)e_H} < z_1 < \frac{H_1^* e_H + a_2 H_2^*}{(H_1^* + H_2^*)e_H},\ 0 < e_H \leq 1 \right\}, \\ J_3 &= \left\{ (z_1,\ e_H) \mid \frac{H_1^* e_H + a_2 H_2^*}{(H_1^* + H_2^*)e_H} < z_1 < 1,\ 0 < e_H \leq 1 \right\}. \end{aligned}$$

A positive equilibrium point E_{++++} exists if and only if system (3) has the parameters which belong to the region J_2. While if system (3) has the parameters in the region J_1, P_1^* becomes negative. It is easy to show that in the region J_1, $\bar{H}_1 > 0$ (i.e. $e_H z_2 - a_1 > 0$) and $\bar{P}_2 > 0$ (i.e. $z_2\bar{H}_1 - z_1\bar{H}_2 > 0$) are always satisfied. Hence a boundary equilibrium point E_{+0++} always exists in J_1. Similarly for the parameters belong to the region J_3, P_2^* becomes negative. Moreover $\tilde{P}_1 > 0$ (i.e. $z_1\tilde{H}_2 - z_2\tilde{H}_1 > 0$) and $\tilde{H}_2 > 0$ (i.e. $e_H z_1 - a_2 > 0$) are always satisfied. Hence another boundary equilibrium point E_{+++0} always exists in J_3.

Theorem 1. (i) Suppose that system (3) satisfies (4), that is, it has the parameters of the region J_2. Then E_{++++} exists and it is globally asymptotically stable, except for the case where the parameters satisfy the following conditions:

$$d_1 = d_2, \quad b_1 P_1^* = b_2 P_2^*. \tag{5}$$

If system (3) satisfies (5), then E_{++++} remains to be neutrally stable.
(ii) If the parameters of system (3) belong to the region J_1 (or J_3), then a boundary equilibrium point E_{+0++} (or E_{+++0}) exists and it is globally asymptotically stable, respectively.

Proof. Define the following Lyapunov functions for each equilibrium point E_{++++}, E_{+0++} or E_{+++0}:

$$\begin{aligned} V(H,\ P) = {} & \frac{H_1^*}{z_1}\left(H_1 - H_1^* - H_1^* \log\frac{H_1}{H_1^*}\right) \\ & + \frac{H_1^*}{f z_1}\left(P_1 - P_1^* - P_1^* \log\frac{P_1}{P_1^*}\right) + \frac{H_2^*}{z_2}\left(H_2 - H_2^* - H_2^* \log\frac{H_2}{H_2^*}\right) \\ & + \frac{H_2^*}{f z_2}\left(P_2 - P_2^* - P_2^* \log\frac{P_2}{P_2^*}\right), \end{aligned}$$

where for E_{+0++} (or E_{+++0}) in the terms for $P_1^* \le 0$ (or $P_2^* \le 0$), we replace $H_1^*,\ H_2^*,\ P_2^*$ (or $H_1^*,\ P_1^*,\ H_2^*$) with $\bar{H}_1,\ \bar{H}_2,\ \bar{P}_2$ (or $\tilde{H}_1,\ \tilde{P}_1,\ \tilde{H}_2$) and the term $P_1 - P_1^* - P_1^* \log P_1/P_1^*$ (or $P_2 - P_2^* - P_2^* \log P_2/P_2^*$) simply with P_1 (or P_2), respectively. These Lyapunov functions $V(H,\ P)$ are defined on the domain

$$\begin{aligned}
\Omega(E_{++++}) &= \{(H_1,\ P_1,\ H_2,\ P_2) \mid P_i > 0,\ H_i > 0\ (i = 1, 2)\},\\
\Omega(E_{+0++}) &= \{(H_1,\ P_1,\ H_2,\ P_2) \mid P_1 \geq 0,\ P_2 > 0\\
&\qquad \text{and } H_i > 0\ (i = 1, 2)\},\\
\Omega(E_{+++0}) &= \{(H_1,\ P_1,\ H_2,\ P_2) \mid P_1 > 0,\ P_2 \geq 0\\
&\qquad \text{and } H_i > 0\ (i = 1, 2)\},
\end{aligned}$$

which correspond to the equilibria E_{++++}, E_{+0++} and E_{+++0}, respectively. Clearly, the solution of (3) starting in $\Omega(E_{++++})$ (or $\Omega(E_{+0++})$, or $\Omega(E_{+++0})$) at $t = 0$ remains there for $t > 0$, respectively.

First we consider E_{++++}. In the region J_2, the time derivative of $V(H,\ P)$ is obtained as:

$$\dot{V} = -\frac{e_H}{H_1 H_2}(H_1 H_2^* - H_1^* H_2)^2 \leq 0. \tag{6}$$

Now we want to find the largest invariant set M of R:

$$R = \{(H_1,\ P_1,\ H_2,\ P_2) \in \Omega(E_{++++}) \mid \dot{V} = 0\}. \tag{7}$$

By the definition of R, we have in R

$$\frac{H_1}{H_2} = \frac{H_1^*}{H_2^*}. \tag{8}$$

Since the time derivative of H_1/H_2 is identically zero in R, we have $\dot{H}_1 H_2 - H_1 \dot{H}_2 = 0$ and

$$b_1 P_1 - b_2 P_2 = \left\{a_1 + e_H\left(\frac{z_1 H_2}{H_1} - z_2\right)\right\} - \left\{a_2 + e_H\left(\frac{z_2 H_1}{H_2} - z_1\right)\right\}.$$

From this and (8), we can derive the following relation

$$b_1 P_1 - b_2 P_2 = b_1 P_1^* - b_2 P_2^*. \tag{9}$$

Hence $b_1 P_1 - b_2 P_2$ becomes constant in R, its time derivative is zero and we can derive

$$b_1 \dot{P}_1 - b_2 \dot{P}_2 = b_1 P_1 (f b_1 H_1 - d_1) - b_2 P_2 (f b_2 H_2 - d_2) = 0. \tag{10}$$

Here if $fb_1H_1 - d_1 = 0$, then $H_1 = H_1^*$ and $H_2 = H_2^*$, $P_1 = P_1^*$ and $P_2 = P_2^*$ are satisfied by (8) and (3), so we obtain $R = \{(H_1^*,\ P_1^*,\ H_2^*,\ P_2^*)\}$ and it is an invariant set. Therefore by LaSalle's Theorem [6], the positive equilibrium point is globally asymptotically stable.

While if $fb_1H_1 - d_1 \neq 0$, by (8) and the definition of H_i^* $(i = 1, 2)$, (10) can be transformed as $(fb_1H_1 - d_1)(b_1P_1 - b_2P_2d_2/d_1) = 0$. As $fb_1H_1 - d_1 \neq 0$, we analyze the following condition

$$b_1P_1 - b_2P_2\frac{d_2}{d_1} = 0. \tag{11}$$

We shall consider the following three cases

$$\begin{cases} \text{(i)} & : \ d_1 \neq d_2 \\ \text{(ii)} & : \ d_1 = d_2 = \delta,\ b_1P_1^* \neq b_2P_2^* \\ \text{(iii)} & : \ d_1 = d_2 = \delta,\ b_1P_1^* = b_2P_2^*. \end{cases}$$

Case (i): From (9) and (11) we can derive the simultaneous equations as $b_1d_1P_1 - b_2d_2P_2 = 0,\ b_1P_1 - b_2P_2 = b_1P_1^* - b_2P_2^*$. Since $d_1 \neq d_2$, they have a unique non-trivial constant solution $(P_1,\ P_2)$ and $\dot{P}_1 = \dot{P}_2 = 0$ implies $H_i = H_i^*$ $(i = 1, 2)$ by (8). Hence a solution of system (3) is only an equilibrium point E_{++++}.

Case (ii): From (9) we have $b_1P_1 - b_2P_2 = b_1P_1^* - b_2P_2^* \neq 0$, which gives a contradiction to (11). So there are no solutions P_i which satisfy case (ii).

Case (iii): If condition (iii) is satisfied, by (11) we have $b_1P_1 = b_2P_2$. So the largest invariant set in R is contained in the following subset M' of R:

$$M' = \{(H_1,\ P_1,\ H_2,\ P_2) \in \Omega(E_{++++}) \mid b_1H_1 = b_2H_2,\ b_1P_1 = b_2P_2\}.$$

In M' we define $u(t)$ and $v(t)$ as follows

$$\begin{cases} b_1H_1(t) = b_2H_2(t) = u(t), \\ b_1P_1(t) = b_2P_2(t) = v(t). \end{cases}$$

By substituting $H_1 = u/b_1,\ P_1 = v/b_1,\ H_2 = u/b_2$ and $P_2 = v/b_2$ into system (3), and putting $\alpha = (a_1 b_2 + a_2 b_1)/(b_1 + b_2)$ we obtain

$$\begin{cases} \dot{u}(t) &= (\alpha - v)u, \\ \dot{v}(t) &= (fu - \delta)v. \end{cases}$$

It is a pure L-V equation and its trajectory is invariant. So if condition (iii) is satisfied, the dynamics of system (3) is given by the pure L-V dynamics.

Now let us consider E_{+0++} or E_{+++0}. The time derivative of $V(H,\ P)$ is given by

$$\begin{aligned} \dot{V} &= \frac{\bar{H}_1}{fz_1}(fb_1\bar{H}_1 - d_1)P_1 - \frac{e_H}{H_1H_2}(H_1\bar{H}_2 - \bar{H}_1H_2)^2 \le 0 \quad \text{in } J_1, \\ \Bigg(\text{or } \dot{V} &= \frac{\tilde{H}_2}{fz_2}(fb_2\tilde{H}_2 - d_2)P_2 - \frac{e_H}{H_1H_2}(H_1\tilde{H}_2 - \tilde{H}_1H_2)^2 \le 0 \quad \text{in } J_3\Bigg). \end{aligned}$$

It is easy to show that $fb_1\bar{H}_1 - d_1 < 0$ in J_1 and $fb_2\tilde{H}_2 - d_2 < 0$ in J_3. So we can derive that the largest invariant set of $R = \{(H_1,\ P_1,\ H_2,\ P_2) \in \Omega(E_{+0++}) \mid \dot{V} = 0\}$, (or $R = \{(H_1,\ P_1,\ H_2,\ P_2) \in \Omega(E_{+++0}) \mid \dot{V} = 0\}$), is E_{+0++} (or E_{+++0}) and it's clearly invariant. By LaSalle's Theorem, the proof of global stability of E_{+0++} (or E_{+++0}) is completed.

3 Model with Predator Migration

We now prevent prey movement ($e_H = 0$), so system (1) becomes as follows:

$$\begin{aligned} \dot{H}_1 &= a_1H_1 - b_1H_1P_1, \\ \dot{P}_1 &= fb_1H_1P_1 - d_1P_1 - e_PP_1 + q_1e_P(P_1 + P_2), \\ \dot{H}_2 &= a_2H_2 - b_2H_2P_2, \\ \dot{P}_2 &= fb_2H_2P_2 - d_2P_2 - e_PP_2 + q_2e_P(P_1 + P_2). \end{aligned} \tag{12}$$

There exist four nonnegative equilibrium points:

$$\begin{aligned} E_{0000} &= (0,\ 0,\ 0,\ 0) \quad &, \quad E_{++0+} &= (\bar{H}_1,\ \bar{P}_1,\ 0,\ \bar{P}_2), \\ E_{++++} &= (H_1^*,\ P_1^*,\ H_2^*,\ P_2^*) \quad &, \quad E_{0+++} &= (0,\ \tilde{P}_1,\ \tilde{H}_2,\ \tilde{P}_2), \end{aligned}$$

where $\bar{H}_1,\ \bar{H}_2,\ \bar{P}_2,\ H_1^*,\ P_1^*,\ H_2^*,\ P_2^*,\ \tilde{P}_1,\ \tilde{H}_2$ and $\tilde{P}_2$ are given by

$$\begin{cases} \bar{H}_1 &= d_1/(fb_1) + e_P(q_2\bar{P}_1 - q_1\bar{P}_2)/(fb_1\bar{P}_1), \\ \bar{P}_1 &= a_1/b_1, \\ \bar{P}_2 &= e_Pq_2\bar{P}_1/(d_2 + e_Pq_1), \end{cases}$$

$$\begin{cases} H_1^* &= d_1/(fb_1) + e_P(q_2P_1^* - q_1P_2^*)/(fb_1P_1^*), \\ P_1^* &= a_1/b_1, \\ H_2^* &= d_2/(fb_2) + e_P(q_1P_2^* - q_2P_1^*)/(fb_2P_2^*), \\ P_2^* &= a_2/b_2, \end{cases}$$

$$\begin{cases} \tilde{P}_1 &= e_Pq_1\tilde{P}_2/(d_1 + e_Pq_2), \\ \tilde{H}_2 &= d_2/(fb_2) + e_P(q_1\tilde{P}_2 - q_2\tilde{P}_1)/(fb_2\tilde{P}_2), \\ \tilde{P}_2 &= a_2/b_2. \end{cases}$$

E_{++++} exists if and only if

$$\frac{e_PP_1^* - d_2P_2^*}{(P_1^* + P_2^*)e_P} < q_1 < \frac{P_1^*(d_1 + e_P)}{(P_1^* + P_2^*)e_P}. \tag{13}$$

The Fig.2 illustrates this condition. Here, we define the regions as follows:

$$\begin{aligned} K_1 &= \left\{(q_1,\ e_P) \mid 0 < q_1 < \frac{e_PP_1^* - d_2P_2^*}{(P_1^* + P_2^*)e_P},\ 0 < e_P \leq 1\right\}, \\ K_2 &= \left\{(q_1,\ e_P) \mid \frac{e_PP_1^* - d_2P_2^*}{(P_1^* + P_2^*)e_P} < q_1 < \frac{P_1^*(d_1 + e_P)}{(P_1^* + P_2^*)e_P},\ 0 < e_P \leq 1\right\}, \\ K_3 &= \left\{(q_1,\ e_P) \mid \frac{P_1^*(d_1 + e_P)}{(P_1^* + P_2^*)e_P} < q_1 < 1,\ 0 < e_P \leq 1\right\}. \end{aligned}$$

A positive equilibrium point E_{++++} exists if and only if system (12) has the parameters which belong to the region K_2. While if system (12) has the parameters of the region K[illegible], H_2^* becomes negative and a boundary equilibrium point E_{++0+} exists. Similarly when the parameters of system (12) belong to the region K_3, H_1^* becomes negative and another boundary equilibrium point E_{0+++} exists in K_3. The stability of the equilibrium points are given by the following theorem, which was proved by Kawasaki and Teramoto [2] for the model with N patches and $q_i = 1/2$.

Theorem 2. (i) Suppose that system (12) has the parameters of the region K_2. Then E_{++++} exists and it is globally asymptotically stable, except for the case where the parameters satisfy the following conditions:

$$a_1 = a_2, \qquad b_1 H_1^* = b_2 H_2^*. \tag{14}$$

If system (12) satisfies (14), then E_{++++} remains to be neutrally stable. (ii) If the parameters of system (12) belong to the region K_1 (or K_3), then E_{++0+} (or E_{0+++}) exists and is globally asymptotically stable, respectively.

4 Effects of Migration and Spatial Heterogeneity

For the prey (or predator) migration model, if migration occurs not so frequently (that is, the emigration rate of prey (or predator) e_H (or e_P) is smaller than $\min\{a_1,\ a_2\}$ (or $\min\{d_1P_1^*/P_2^*,\ d_2P_2^*/P_1^*\}$) respectively), a positive equilibrium point exists under any prey's (or predator's) preference of immigration to one of two patches. That is, the coexistence of all species is possible for any value of z_i, q_i $(i = 1, 2)$. While if migration occurs frequently (that is, e_H (or e_P) is bigger than $\min\{a_1,\ a_2\}$ (or $\min\{d_1P_1^*/P_2^*,\ d_2P_2^*/P_1^*\}$) respectively), the prey's (or predator's) extreme preference for one patch over another patch extinguishes a positive equilibrium point. So, in one of two patches, the predator (or prey) goes extinct.

On the other hand, near the asymptotic line of both sides of condition (4) (or (13)):

$$z_1 = \frac{H_1^*}{H_1^* + H_2^*}, \tag{15}$$

$$\left(\text{or } q_1 = \frac{P_1^*}{P_1^* + P_2^*}\right), \tag{16}$$

the predators in both patches can coexist with prey under any emigration rate e_H (or e_P) of prey (or predators). Here note that H_i^* (or P_i^*) $(i = 1, 2)$ represent not only positive equilibrium values of prey (or predators) population in both patches but also the time averages of the prey (or predators) population. Hence (15) (or (16)) implies the following: when the time average of prey (or predator) in patch 1 is larger than one in patch 2 ($H_1^* > H_2^*$ (or $P_1^* > P_2^*$)), the best immigration rate z_1 (or q_1) of (15) (or (16)) becomes larger than 1/2. For this case, the prey (or predators) should aggregate to patch 1 preferentially ($z_1 > z_2$ (or $q_1 > q_2$)). While the prey's (or predator's) excessive preferential immigration to one of patches derives the extinction of the predators (or prey) in another patch. That is, proper emigration rate of prey (or predators) and the aggregation of prey (or predators) are sufficient to promote the diversity of species. It is important to note that, spatial heterogeneity of patches, such as $d_1 \neq d_2$ (or $a_1 \neq a_2$) changes the property of a positive equilibrium point from neutrally stable one to global asymptotically stable one.

Further note that, prey (or predator) in at most one patch can be extinct under any emigration rate for the predator (or prey) and any immigrating rate of the predator (or prey). That is, both prey and predator can survive in at least one patch. This property does not always hold for the model with intraspecific competition among prey, which will be considered in [5]. It seems strange that E_{++0+} (or E_{0+++}) can be globally asymptotically stable. That is, the predator can maintain its number positive in patch 2 (or 1) without help of prey there. Note that this is possible only for predator's migration model and the predator in patch 1 (or 2) can keep its number in patch 2 (or 1) positive with large q_2 (or q_1) (see K_1 (or K_3) in Fig.2), that is, with a strong preference of the immigration to patch 2 (or 1) over patch 1 (or 2).

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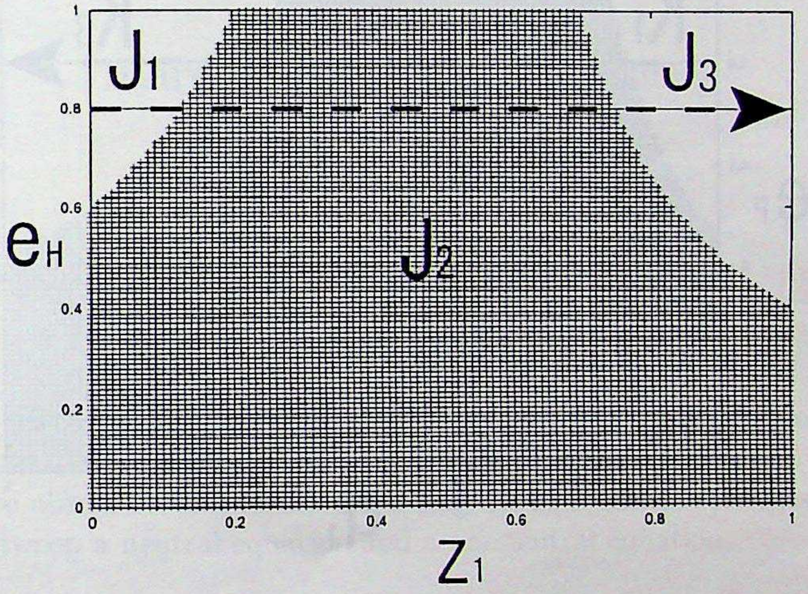

Figure 1: The existence condition on a positive equilibrium point of system (3) as a function of two parameters, the rate of prey immigrating into patch 1 (z_1) and the emigration rate (e_H). The other parameters are fixed ($a_1 = 0.6,\ a_2 = 0.4,\ b_1 = 2.0,\ b_2 = 2.0,\ d_1 = 1.0,\ d_2 = 1.0,\ f = 1.0$). In the hatched region J_2, condition (4) is satisfied and E_{++++} exists. Further, E_{+0++} (or E_{+++0}) exists in J_1 (or J_3), respectively.

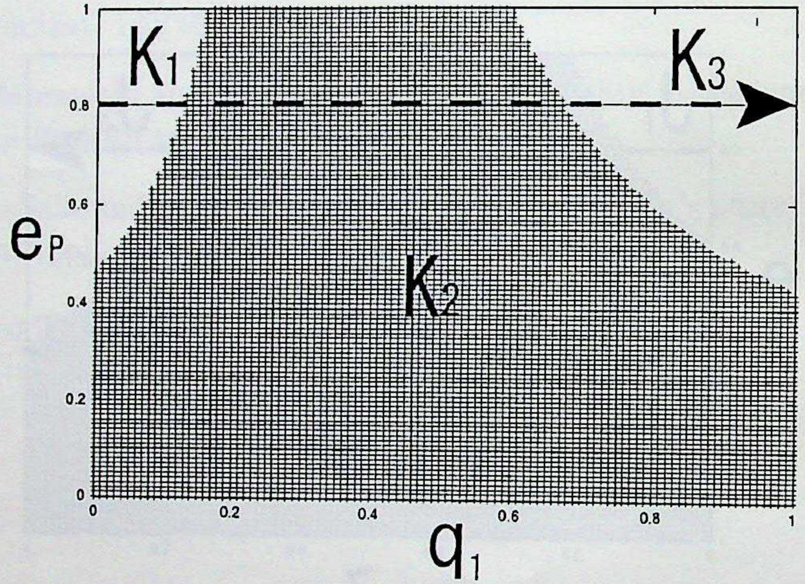

Figure 2: The existence condition of a positive equilibrium point of system (12) as a function of two parameters, the immigrating rate of predator to patch 1 (q_1) and the emigration rate (e_P). The other parameters are fixed ($a_1 = 0.6,\ a_2 = 1.4,\ b_1 = 1.0,\ b_2 = 1.0,\ d_1 = 1.0,\ d_2 = 0.2,\ f = 1.0$). In the region K_2, E_{++++} exists. E_{++0+} (or E_{0+++}) exists in K_1 (or K_3), respectively.

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Oscillation of a First Order Neutral Equation and a Second Order Nonneutral Equation

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(Submitted by: J. Wu)

Abstract

In this paper, we establish several comparison theorems on the oscillation of a neutral equation and a nonneutral equation. We also obtain several results regarding the equivalence of oscillation between a neutral equation and a nonneutral equation.

1 Introduction

Consider the following first order neutral differential equation (NDE for short)

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + f_1(t, x(g_1(t)), \cdots, x(g_n(t))) = 0 \qquad (1)$$

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and the second order non NDE

$$\frac{d^2}{dt^2}y(t) + f_2(t, y(h_1(t)), \cdots, y(h_n(t))) = 0, \tag{2}$$

where the delay τ is a positive constant, the following hypotheses are assumed:

(H_1) $P : [t_0, \infty) \to R^+$ is continuous;

(H_2) $g_i, h_i : [t_0, \infty) \to R$, $i = 1, 2, \cdots, n$, are continuous and $\lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} h_i(t) = \infty$;

(H_3) $f_i : [t_0, \infty) \times R^n \to R$, $i = 1, 2$, are continuous and satisfy $z_1 f_i(t, z_1, \cdots, z_n) > 0$ for $z_1 z_j > 0$, $j = 1, 2, \cdots, n$.

Our aim is to establish some comparison theorems concerning the oscillation behaviour of NDE (1) and non NDE (2).

In particular, we show that under some additional hypotheses on τ, P, g and f, there exists certain equivalence relations for the oscillation between NDE (1) and non NDE (2).

By a solution of Eq(1), we mean a continuous function $x : (t_0, \infty) \to R$ such that $x(t) - p(t)x(t-\tau)$ is continuously differentiable and satisfies (1) for all $t > t_0$. For the existence, uniqueness and continuous dependence of solution of Eq.(1), we refer to [3].

A solution of (1) (or (2)) is said to oscillate if it has arbitrary large zeros, nonoscillate if otherwise.

2 Main Results

In the sequel, we will assume that conditions (H_1)-(H_3) are satisfied. Define

$$P_0(t) = 1 \text{ for } t \geq t_0$$

and for $k = 1, 2, \cdots$

$$P_k(t) = \prod_{i=0}^{k-1} P(t + i\tau), \quad t \geq t_0.$$

The following Theorem 1 gives a sufficient condition to ensure that the oscillation of all solutions of Eq.(2) implies the same for Eq(1).

Theorem 1. Assume that there exist a positive integer N and a $t^* \geq t_0$ such that

$$P_k(t) \geq 1 \quad \text{for} \quad t \geq t_0, \quad k = N,\ N+1, \cdots \tag{3}$$

and

$$\sum_{k=N}^{\infty} \frac{1}{P_k(t^*)} = \infty. \tag{4}$$

Suppose also that there exists $\sigma \geq 0$ such that

$$g_i(t) + \sigma \geq h_i(t) \quad \text{for} \quad t \geq t_0, \quad i = 1, 2, \cdots, n. \tag{5}$$

Finaly assume that

$$\frac{1}{\tau}|f_1(t, z_1, \cdots, z_n)| \geq |f_2(t, z_1, z_2, \cdots, z_n)| \tag{6}$$

for $t \geq t_0$ and $z_1 z_j > 0, \quad j = 1, 2, \cdots, n.$

and that

$$f_2(t, z_1, z_2, \cdots, z_n) \text{ is nondecreasing in each } z_j \text{ for } z_1 z_j > 0, \tag{7}$$

$j = 1, 2, \cdots, n.$

Then oscillation of all solutions of Eq.(2) implies the oscillation of all solutions of Eq(1).

In order to prove this theorem, we need the following lemma:

Lemma 1. Assume that there exists a $t^* \geq t_0$ such that

$$P_k(t^*) > 0 \text{ for } k = 1, 2, \cdots \tag{8}$$

and that (4) is satisfied. Let $x(t)$ be a positive solution of the differential inequality

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + f_1(t, x(g_1(t)), x(g_2(t)), \cdots, x(g_n(t))) \leq 0, \tag{9}$$

and set

$$Z(t) = x(t) - P(t)x(t-\tau). \tag{10}$$

Then eventually

$$Z(t) > 0. \tag{11}$$

Proof. From (9) and (10), we have

$$Z'(t) \leq -f_1(t, x(g_1(t)), \cdots, x(g_n(t))) < 0$$

if (11) is not satisfied, then we have $Z(t) < 0$. Thus there exist $c > 0$ and a sufficiently large $t_1 > t_0$ such that $Z(t) \leq -c$ for $t \geq t_1$. i.e.,

$$x(t) \leq -c + P(t)x(t-\tau) \text{ for } t \geq t_1.$$

Choose a positive integer n_0 such that $n_0\tau + t^* \geq t_1$. Then

$$\begin{aligned}\frac{x(t^* + (n_0+k)\tau)}{P_{n_0+k+1}(t^*)} &\leq -\frac{c}{P_{n_0+k+1}(t^*)} \\ &+ \frac{P(t^* + (n_0+k)\tau)x(t^* + (n_0+k-1)\tau)}{P_{n_0+k+1}(t^*)}.\end{aligned}$$

That is, in view of (4)

$$\frac{x(t_* + (n_0+k)\tau)}{P_{n_0+k+1}(t^*)} \leq -\frac{c}{P_{n_0+k+1}(t^*)} + \frac{x(t^* + (n_0+k-1)\tau)}{P_{n_0+k}(t^*)},$$

which yields

$$\frac{x(t_* + (n_0+k)\tau)}{P_{n_0+k+1}(t^*)} \leq -c\sum_{i=2}^{k+1} \frac{1}{P_{n_0+i}(t^*)} + \frac{x(t_* + n_0\tau)}{P_{n_0+1}(t^*)}.$$

Let $k \to \infty$, then by (4), we have

$$\frac{x(t^* + (n_0+k)\tau)}{P_{n_0+k+1}(t^*)} \to -\infty.$$

This contradicts the positivity of $x(t)$, and completes the proof.

Remark 1. In Lemma 1, condition (4) can be replaced by the condition that $P(t)$ has arbitrarily large zeros, i.e. $P(t)$ is oscillating. However, when $P(t)$ is positive, condition (4) is weaker than that the condition

$$P(t^* + k\tau) \leq 1 \text{ for } k = 0, 1, 2, \cdots$$

which is frequently used in the literature (see [5]).

Remark 2. Similar lemma can be established for the negative solution of the differential inequality

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + f_1(t, x(g_1(t)), \cdots, x(g_n(t))) \geq 0.$$

Proof of Theorem 1. Assume, for the sake of contradiction, that Eq.(1) has a nonoscillatory solution $x(t)$, we will only consider the case that $x(t)$ is positive as the proof for the case when $P(t)$ is negative is similar. Let $Z(t)$ be defined as (10). Then (11) is satisfied, and so we have

$$x(t) > P(t)x(t-\tau),$$

which, together with (3) and (4), implies

$$x(t) > P_N(t-(N-1)\tau)x(t-N\tau) \geq x(t-N\tau).$$

Therefore, there exists $t_1 > t_0$ and a $c > 0$ such that,

$$x(t) \geq c \text{ and } x(g_i(t)) > 0 \text{ for } t \geq t_1, \quad i = 1, 2, \cdots, n.$$

Choose a $t_2 \geq t_1 + (N+1)\tau + \sigma$ such that $g_i(t) \geq t_1 + (N+1)\tau$ for $t \geq t_2$, $i = 1, 2, \cdots, n$. By integrating (1) from $g_i(t)$ to ∞, we obtain

$$x(g_i(t)) \geq P(g_i(t))x(g_i(t)-\tau) + \int_{g_i(t)}^{\infty} f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds$$

for $t \geq t_2, \quad i = 1, 2, \cdots, n$, which implies

$$\begin{aligned} x(g_i(t)) \quad \geq \quad & P_{m(g_i(t))}(g_i(t) - (m(g_i(t)-1)\tau)x(g_i(t) - m(g_i(t))\tau)) \\ & + \sum_{k=0}^{m(g_i(t))-1} P_k(g_i(t)-(k-1)\tau) \times \\ & \times \int_{g_i(t)-k\tau}^{\infty} f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds \end{aligned}$$

for $t \geq t_2, \quad i = 1, 2, \cdots, n$, where $m(g_i(t)) \geq N$, $i = 1, 2, \cdots, n$ are the positive integers such that

$$t_1 \leq g_i(t) - m(g_i(t))\tau < t_1 + \tau, \quad i = 1, 2, \cdots, n.$$

Using (3) and $x(g_i(t) - m(g_i(t))\tau) \geq c$ and $\int_u^\infty f(s, x(g_1(s), \cdots, x(g_n(s)))ds$ is decreasing to u, then

$$\begin{aligned}
x(g_i(t)) &\geq c + \sum_{k=N}^{m(g_i(t))-1} \int_{g_i(t)-k\tau}^{\infty} f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds \\
&\geq c + \sum_{k=N}^{m(g_i(t))-1} \frac{1}{\tau} \int_{g_i(t)-k\tau}^{g_i(t)-(k-1)\tau} \int_u^\infty f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds\, du \\
&\geq c + \frac{1}{\tau} \int_{t_1+2\tau}^{g_i(t)-(N-1)\tau} \int_u^\infty f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds\, du \\
&\geq c + \frac{1}{\tau} \int_{t_1+2\tau+(N+1)\tau+\sigma}^{g_i(t)-(N-1)\tau+(N+1)\tau+\sigma} \int_u^\infty f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds\, du
\end{aligned}$$

for $t \geq t_2, \quad i = 1, 2, \cdots, n$, and so

$$x(g_i(t)) \geq c + \frac{1}{\tau} \int_{t_2}^{g_i(t)+\sigma} \int_u^\infty f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds\, du \tag{12}$$

for $t \geq t_2, \quad i = 1, 2, \cdots, n$. Set

$$\omega(t) = c + \int_{t_2}^{t} \int_u^\infty f_2(s, x(g_1(s)), \cdots, x(g_n(s)))ds\, du. \tag{13}$$

Choose $t_3 \geq t_2$ such that $h_i(t) \geq t_2$ for $t \geq t_3$, $i = 1, 2, \cdots, n$.

In view of (5), (12) and (13), we have

$$0 < \omega(h_i(t)) \leq x(g_i(t)) \text{ for } t \geq t_3, \quad i = 1, 2, \cdots, n. \tag{14}$$

Let C_B denote the Banach space of all bounded and continuous functions $v : [t_0, \infty) \to R$ equipped with the sup norm $\|\ \|$. Defined a bounded, closed and convex subset of C_B by

$$V = \{v \in C_B : 0 \leq v(t) \leq 1, \text{ for } t \geq t_0\}$$

and define the operator S on V as follows:

$$(Sv)(t) = \begin{cases} \frac{1}{w(t)}\Big[c + \int_{t_3}^{t} \int_u^\infty f_2(s, v(h_1(s))w(h_1(s)), \cdots, \\ \qquad\qquad v(h_n(s))w(h_n(s)))dsdu\Big] \quad t \geq t_3 \\ (Sv)(t_3), \quad t_0 \leq t \leq t_3. \end{cases} \tag{15}$$

Now we will apply Schauder fixed point theorem to derive that S has a fixed point in V.

First, for any $v \in V$ and $t \geq t_3$, we see from (13)-(15) that

$$(Sv)(t) \leq \frac{1}{w(t)}\left[c + \frac{1}{\tau}\int_{t_3}^{t}\int_{u}^{\infty} f_2(s, x(g_1(s)), x(g_n(s)))ds\ du\right] \leq 1,$$

which shows that S maps V into itself.

Next we suppose $\lim\limits_{k\to\infty} ||v_k - v|| = 0$ with $v_k \in V$. Then $v \in V$, It follows from (13) and (15) that

$$\begin{aligned}
&|(Sv_k)(t) - (Sv)(t)| \leq \\
&\quad \frac{1}{c}\int_{t_3}^{t}\int_{u}^{\infty} |f_2(s, v_k(g_1(s))w(h_1(s)), \cdots, v_k(g_n(s))w(h_n(s))) \\
&\quad -f_2(s, v(g_1(s))w(h_1(s)), \cdots, v(g_n(s))w(h_n(s)))|ds\ du, \quad t \geq t_3.
\end{aligned}$$

It follows from Lebesgue's dominated convegence theorem that $\lim\limits_{k\to\infty} ||Sv_k - Sv|| = 0$, i.e, S is continuous on V.

Finaly, differentiating both sides of (15) we get

$$\begin{aligned}
\frac{d}{dt}(Sv)(t) = \frac{1}{w(t)}\Big[&-(Sv)(t)\int_{t}^{\infty} f_2(s, x(g_1(s)), \cdots, x(g_n(s)))ds \\
&+\int_{t}^{\infty} f_2(s, v(h_1(s))w(h_1(s)), \cdots, v(h_n(s))w(h_n(s)))ds,\Big] t \geq t_3
\end{aligned}$$

which implies $\left|\frac{d}{dt}(Sv)(t)\right| \leq M$ for any $v \in V$ and $t \geq t_0$, here M is a positive constant by

$$\begin{aligned}
M &= \frac{1}{c}\Big[\int_{t_3}^{\infty} f_2(s, x(g_1(s)), \cdots, x(g_n(s)))ds \\
&+ \int_{t_3}^{\infty} f_2(s, w(h_1(s)), \cdots, w(h_n(s)))ds.\Big]
\end{aligned}$$

Thus $S(V)$ is relatively compact in the Banach space C_B.

Therefore, there exists a $v \in V$ such that $Sv = v$. Set $y(t) = v(t)w(t)$. Then it is easy to see that $y(t)$ is a positive solution of Eq.(2) on $[t_3, \infty)$. This is a contradition and the proof of the theorem is completed.

The following result provides sufficient conditions for the inverse conclusion of Theorem 1.

Theorem 2. Assume that

$$P(t) \leq 1 \quad \text{for} \quad t \geq t_0 \tag{16}$$

and

$$g_i(t) < t, \quad i = 1, 2, \cdots, n. \tag{17}$$

Suppose also that there exists a $\sigma > 0$ such that

$$g_i(t) \leq h_i(t) + \sigma \quad \text{for} \quad t \geq t_0, \qquad i = 1, 2, \cdots, n \tag{18}$$

Finally assume that

$$\tau |f_2(t, z_1, \cdots, z_n)| \geq |f_1(t, z_1, \cdots, z_n)| \tag{19}$$

for $t \geq t_0$ and $z_1 z_i > 0$, $i = 1, 2, \cdots, n$ and that

$$f_1(t, z_1, z_2, \cdots, z_n) \text{ is nondecreasing in each } z_j \text{ for } z_1 z_j > 0, \tag{20}$$

$j = 1, 2, \cdots, n$.

Then all solution of Eq.(1) oscillate implies that all solutions of Eq.(2) also oscillate.

Proof. Otherwise, Eq.(2) has a positive solution $y(t)$. Then $y''(t) < 0$ and hence $y'(t) > 0$. By integrating (2) from t to ∞, we have

$$y'(t) \geq \frac{1}{\tau} \int_t^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds. \tag{21}$$

Choose a $b > 0$ and $t_1 > t_0$ such that $y(t - \sigma) \geq b$ for $t \geq t_1$, $y(h(t)) > 0$ for $t \geq t_1 - \sigma$ and

$$\int_{t_1}^\infty f_2(s, y(h_1(s)), \cdots, y(h_n(s)))ds \leq \frac{b}{2}. \tag{22}$$

Let $t_2 \geq t_1$, such that $h_i(t) \geq t_1 - \sigma + \tau$ and $g_i(t) \geq t_1 + \tau \;\; i = 1, 2, \cdots, n$. Integrting (21) from $t_1 - \sigma$ to $h_i(t)$, we get

$$y(h_i(t)) \geq b + \frac{1}{\tau} \int_{t_1 - \sigma}^{h_i(t)} \int_u^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds\, du$$

$$\begin{aligned}
&\geq b + \frac{1}{\tau}\int_{t_1}^{h_i(t)+\sigma}\int_u^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds\,du \\
&\geq b + \\
&+\frac{1}{\tau}\sum_{k=0}^{m(h_i(t)+\sigma)-1}\int_{g_i(t)-k\tau-\tau}^{g_i(t)-k\tau}\int_u^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds\,du \\
&\geq b + \sum_{k=0}^{m(g_i(t))-1}\int_{g_i(t)-k\tau}^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds \qquad (23)
\end{aligned}$$

$t \geq t_2, \quad i = 1, 2, \cdots, n$, where $m(g_i(t))$ denote the postive integer such that $t_1 + \tau > g_i(t) - m(g_i(t))\tau \geq t_1$, for $t_1 \leq t \leq t_1 + \tau$, set

$$\varphi(t) = \frac{t - t_1}{\tau}\int_{t_1+\tau}^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds + \frac{b}{2} \qquad (24)$$

and define

$$Z(t) = \begin{cases} \varphi(t) & t_1 \leq t \leq t_1 + \tau \\ \varphi(t - m(t)\tau) & \\ \quad + \sum\limits_{k=0}^{m(t)-1} \int_{t-k\tau}^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s)))ds & t \geq t_1 + \tau. \end{cases}$$

Then, it is easy to show that $Z(t)$ is continuous and positive on $[t_1, \infty)$ and satisfies

$$Z(t) = Z(t-\tau) + \int_t^\infty f_1(s, y(h_1(s)), \cdots, y(h_n(s))ds, \quad t \geq t_1 + \tau. \quad (25)$$

It follows from (19), (22) and (24) that

$$0 < \varphi(t) \leq b \quad \text{for} \quad t_1 \leq t \leq t_1 + \tau.$$

In view of (23), it is easy to prove that

$$Z(g_i(t)) \leq y(h_i(t)) \quad \text{for} \quad t \geq t_2, \quad i = 1, 2, \cdots, n. \qquad (26)$$

Now we define a set of functions

$$W = \{w \in C([t_1, \infty), R^+) : 0 \leq w(t) \leq 1 \ \text{ for } \ t \geq t_1\}$$

and define a mapping S on W as

$$(Sw)(t) = \begin{cases} \frac{1}{Z(t)}\Big[P(t)w(t-\tau)Z(t-\tau) + \int_t^\infty f_1(s, w(g_1(s))Z(g_1(s)), \cdots, \\ \qquad\qquad w(g_n(s))Z(g_n(s)))ds\Big] \quad t \ge t_1 + m \\ \frac{t-t_1}{m}(Sw)(t_1+m) + 1 - \frac{t-t_1}{m} \quad t_1 \le t \le t_1 + m \end{cases} \tag{27}$$

where $m = \max\{\tau, t_2 - t_1\}$. Then for any $w \in W$ and $t \ge t_1 + m$, from (16), (25)-(27) we have

$$(Sw)(t) \le \frac{1}{Z(t)}\left[Z(t-\tau) + \int_t^\infty f(s, y(h_1(s)), \cdots, y(h_n(s)))ds\right] = 1,$$

which implies that S maps W into itself.

Now we define a sequnce $\{w_k(t)\}$ in W:

$$\begin{aligned} &w_0(t) = 1, \quad t \ge t_1; \\ &w_{k+1}(t) = (Sw_k)(t), \quad k = 0, 1, 2, \cdots. \end{aligned}$$

Then it is easy to see by a simple induction that

$$0 \le w_{k+1}(t) \le w_k(t) \le 1 = w_0(t), \quad t \ge t_1, \quad k = 0, 1, 2, \cdots$$

which implies that

$$w(t) = \lim_{k\to\infty} w_k(t) \quad t \ge t_1$$

exists and it follows from Lebesgue monotone convegence theorem that $w(t)$ satisfies

$$w(t) = \begin{cases} \frac{1}{Z(t)}\Big[P(t)w(t-\tau)Z(t-\tau) + \int_t^\infty f_1(s, w(g_1(s))Z(g_1(s)), \cdots, \\ \qquad\qquad w(g_n(s))Z(g_n(s)))ds\Big] \quad t \ge t_1 + m \\ \frac{t-t_1}{m}w(t_1+m) + 1 - \frac{t-t_1}{m}, \quad t_1 \le t \le t_1 + m. \end{cases}$$

Set $x(t) = w(t)z(t)$. Then $x(t)$ satisfies

$$x(t) = p(t)x(t-\tau) + \int_t^\infty f_1(s, x(g_1(s)), \cdots, x(g_n(s)))ds, \quad t \ge t_1 + m$$

and $x(t) > 0$ for $t_1 \le t \le t_1 + m$, In view of (17), it follows from the method of steps that $x(t) > 0$ for $t \ge t_1 + m$, In view of (17), it follows from the method of steps that $x(t) > 0$ for $t \ge t_1 + m$, Therefore, $x(t)$ is a positive solution of Eq.(1). This contradicts the hypotheses that all solution of Eq.(1) oscillate and the proof of the theorem is completed.

The following equivalence results can be obtained by using Theorem 1 and 2.

Deduction. Suppose that (H_1)-(H_3), (7), (20) hold. Futhermore, assume (A) $\frac{1}{\tau} f_1(t, z_1, \cdots, z_n) = f_2(t, z_1, \cdots, z_n) \quad z_1 z_i > 0, \quad i = 1, 2, \cdots, n;$

(B) there exist a positive σ, such that

$$|g_i(t) - h_i(t)| \le \sigma, \quad t \ge t_0, \quad i = 1, 2, \cdots, n.$$

Then the following statements are equal

(i) all solution of Eq.(1) oscillate;

(ii) all solution of Eq.(2) oscillate.

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On the Definitions of the Nonlinear Limit-Point/Limit-Circle Properties

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Abstract

In this paper, two definitions of the nonlinear differential equation with quasiderivatives $y^{(n)} = r(t) f(y^{[0]}, \ldots, y^{[n-1]})$ being of the nonlinear limit-point/limit-circle type are discussed and the relationships between them are investigated.

1 Introduction

Consider the n-th order nonlinear differential equation

$$y^{[n]} = r(t)\, f(y^{[0]}, y^{[1]}, \ldots, y^{[n-1]}), \tag{1}$$

where $n \geq 2$, $y^{[i]}$ is the i-th quasiderivative of y defined by

$$y^{[0]} = y, \quad y^{[i]} = \frac{1}{a_i(t)}(y^{[i-1]})', \ i = 1, 2, \ldots, n-1, \quad y^{[n]} = (y^{[n-1]})', \tag{2}$$

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[0]

the functions $a_i : R_+ \to R$, $i = 1, \dots, n-1$, and $f : R^n \to R$ are continuous, $r \in L_{\text{loc}}(R_+)$, $a_i > 0$, $i = 1, \dots, n-1$, $f(0, 0, \dots, 0) = 0$, and

$$x_1 f(x_1, \dots, x_n) \geq 0 \quad \text{on} \quad R^n. \tag{3}$$

A function y defined on $J = [t_y, b) \subset R_+$ is said to be a *solution* of Eq. (1) if $y^{[n-1]}$ is absolutely continuous on J, y satisfies Eq. (1) for almost all $t \in J$, and either $b = \infty$, or $b < \infty$ and $\sup_{t \in J} |y^{[n-1]}(t)| = \infty$. A solution y is said to be *oscillatory* if there exists a sequence of its zeros tending to b; otherwise, it is called *nonoscillatory.* A solution y is *continuable* if $b = \infty$, and it is *singular* if $b < \infty$. We denote by $[\![a]\!]$ the integer part of the number a.

Remark 1. Note that a solution y that is trivial in a neighborhood of ∞ is oscillatory according to our definition above.

The investigation of solutions of the special case of Eq. (1)

$$y'' = r(t)y \tag{4}$$

belonging to $L^2(R_+)$ has a long history, see e.g., [8]. Recall that Eq. (4) is said to be of the limit-circle type if all its (continuable) solutions belong to $L_2(R_+)$. Otherwise, it is said to be of the limit-point type. The following definition has generalized this concept to Eq. (1), see [2 – 5, 10 – 12].

Definition 1. A continuable solution y of equation (1) is said to be of the (*nonlinear*) *limit-circle type* if it satisfies

$$\int_0^\infty y(t)\, f(y^{[0]}(t), y^{[1]}(t), \dots, y^{[n-1]}(t))\, dt < \infty\,. \tag{5}$$

Otherwise, it is of the (*nonlinear*) *limit-point type*, i.e.,

$$\int_0^\infty y(t)\, f(y^{[0]}(t), y^{[1]}(t), \dots, y^{[n-1]}(t))\, dt = \infty\,. \tag{6}$$

Equation (1) is said to be of the (*nonlinear*) *limit-circle type* if all its continuable solutions satisfy (5), and it is said to be of the (*nonlinear*) *limit-point type* if there exists a continuable solution y for which (6) holds.

Remark 2. Since the trivial solution $y \equiv 0$ is of the limit-circle type, the definition of Eq. (1) being of the limit-circle type does not depend on the existence of such a solution, as is the case with limit-point type solutions.

Note that Definition 1 is not vacuous.

Example 1. The equation

$$y^{(n)} = y$$

is of the limit-point type since $y(t) = e^t$ is a solution of this equation. On the other hand, the equation

$$y''' = f(y),$$

with $f(y) = y$ for $|y| \leq 1$ and $f(y) = \frac{\operatorname{sgn} y}{y^2}$ for $|y| > 1$, is of the limit-circle type (see [2; Theorem 3.3]).

In this paper, two problems will be examined. The first one is devoted to the form of Definition 1, more precisely, how to generalize the limit-point/limit-circle property from Eq. (4) to Eq. (1). The second problem focuses on the fact that in Definition 1 only continuable solutions are considered. However, Eq. (1) may have singular solutions (see [6]). Thus, a more general definition to include both types of solutions is needed. Since our purpose is to study the asymptotic behavior of solutions in a left-hand neighborhood of the right hand end point of the maximal interval of definition, we restrict our attention to solutions y that are defined on $[t_y, T_y) \subset R_+$. Note that according to the definition, y can not be defined for $t \geq T_y$ if $T_y < \infty$.

Definition 2. A solution $y : [t_y, T_y) \to R$ of Eq. (1) is said to be of the *(nonlinear) limit-circle type* if

$$\int_{t_y}^{T_y} y(t)\, f(y^{[0]}(t), \ldots, y^{[n-1]}(t))\, dt < \infty; \tag{7}$$

if

$$\int_{t_y}^{T_y} y(t)\, f(y^{[0]}(t), \ldots, y^{[n-1]}(t))\, dt = \infty, \tag{8}$$

then y it is said to be of the *(nonlinear) limit-point type.* We will say that Eq. (1) is of the *(nonlinear) limit-circle type* if all its solutions satisfy (7), and it will be said to be of the *(nonlinear) limit-point type* if there is at least one solution y satisfying (8).

In the remainder of this paper, we discuss some relationships between these two definitions of the limit-point/limit-circle property.

2 Definition 1

First, we consider the second order nonlinear equation

$$(a(t)\,y')' = r(t) f(y) \tag{9}$$

which is a special case of (1) with $n = 2$ and $a_1 = \frac{1}{a}$, and we will assume that $r \leq 0$ on R_+. Observe that here the right hand side of (9) does not depend on y'. Special cases of (9), namely, the Emden-Fowler equations

$$y'' = -t^\sigma y^\gamma,$$

where $\gamma \neq 1$ is the ratio of odd positive integers, and

$$y'' = r(t)\,y^{2n-1}, \tag{10}$$

where $n = 2, 3, \ldots$, will serve as our motivating prototypes.

As was pointed out in [10], the exact form that integrability results for the second order nonlinear equation (9) should take is not completely obvious. While in the case of equation (10), showing that solutions belong to L^{2n} seems to be the appropriate result, for equation (9), both

$$\int_0^\infty y(t)\, f(y(t))\, dt < \infty \tag{11}$$

and

$$\int_0^\infty F(y(t))\, dt < \infty, \tag{12}$$

where

$$F(y) = \int_0^y f(u)\, du, \tag{13}$$

agree with this choice and also agree with the square integrability of solutions of (4). As the following example shows, for more general classes of equations, conditions (11) and (12) are not equivalent.

Example 2. The function $y(t) = (t+1)^{\frac{1}{2}}$, $t \geq 0$, is a solution of the equation

$$y'' = -\frac{1}{4}(t+1)^{\frac{1}{2}} f(y)$$

where

$$f(x) = \begin{cases} \frac{\operatorname{sgn} x}{x^4}, & \text{for } |x| \geq 1, \\ x, & \text{for } |x| < 1. \end{cases}$$

We see that

$$\int_0^\infty y(t)f(y(t))\,dt < \infty \quad \text{while} \quad \int_0^\infty \int_0^{y(t)} f(u)\,du\,dt = \infty.$$

We next show that (11) appears to be preferable to (12); the basis for this is as follows. We restrict our attention to a special case of Eq. (1) with $a_i \equiv 1$, $i = 1, 2, \ldots, n-1$, namely,

$$y^{(n)} = r(t)\,f(y, y', \ldots, y^{(n-1)}), \tag{14}$$

where $r \in L_{\text{loc}}(R_+)$, $f \in C^0(R^n)$, $\alpha \in \{0, 1\}$,

$$(-1)^\alpha r(t) \geq 0 \text{ on } R_+, \quad \text{and} \quad x_1 f(x_1, \ldots, x_n) \geq 0 \text{ on } R^n. \tag{15}$$

Let y be a continuable solution of Eq. (14). Define $n_0 = [\![\frac{n}{2}]\!]$ and

$$\begin{aligned} E(t) \;=\; & \sum_{i=0}^{n_0-1} (-1)^{\alpha+i} y^{(n-i-1)}(t) y^{(i)}(t) + \frac{1}{2}(-1)^{\alpha+n_0}(n-2n_0)(y^{(n_0)}(t))^2 \\ & +(n-2n_0-1)(-1)^{\alpha+n_0-1}\int_0^t (y^{(n_0)}(s))^2\,ds \end{aligned}$$

for $t \in R_+$. Then, in view of (15), an easy calculation (see [1; Lemma 1.1]) shows that

$$E'(t) = (-1)^\alpha\, r(t)\, f(y(t), \ldots, y^{(n-1)}(t))\, y(t) \geq 0 \text{ a.e. on } R_+.$$

Thus, E is nondecreasing, and (15) yields

$$\int_0^\infty |r(s)|\, y(s)\, f(y(s), \ldots, y^{(n-1)}(s))\,ds < \infty \Leftrightarrow E \text{ is bounded on } R_+. \tag{16}$$

Hence, we see that for $n = 2$, an integral similar to (11) with the weight function $|r|$ has been obtained.

An integral condition similar to (12) can be obtained for the special case of Eq. (14),

$$y^{(n)} = r(t)\,f(y), \tag{17}$$

where $r \in C^1(R_+)$, $f \in C^0(R)$, $\alpha \in \{0, 1\}$, $(-1)^\alpha r(t) \geq 0$ on R_+, $xf(x) \geq 0$ for $x \in R$, and either

$$r' \geq 0 \quad \text{or} \quad r' \leq 0 \quad \text{on} \quad R_+. \tag{18}$$

To see this, define

$$\begin{aligned} E_1(t) &= (-1)^\alpha r(t)\,F(y(t)) + \sum_{i=0}^{n_0-1}(-1)^{i+\alpha+1}y^{(n-i-1)}(t)y^{(i+1)}(t) \\ &\quad +\frac{1}{2}(-1)^{\alpha+n_0}(n-1-2n_0)(y^{(n_0)}(t))^2 \\ &\quad +(n-2n_0)(-1)^{\alpha+n_0-1}\int_0^t [y^{(n_0+1)}(s)]^2\,ds \end{aligned}$$

for $t \in R_+$, where $n_0 = [\![\frac{n}{2}]\!]$ as before and F is given by (13). Then a direct computation yields (see [1; Lemma 3.11] for special values of n)

$$E_1'(t) = (-1)^\alpha r'(t) F(y(t)).$$

Thus, (18) implies E_1 is monotonic and

$$\int_0^\infty |r'(t)|F(y(t))\,dt < \infty \Leftrightarrow E_1 \text{ is bounded on } R_+. \tag{19}$$

In this case, a weighted integral of the form (12) is obtained.

We prefer investigating relation (11) rather than (12) because (16) is valid for a larger class of equations than is covered by (19).

3 Relation Between Definitions 1 and 2

The distinction made here is a result of the co-existence of both nontrivial continuable solutions and singular solutions. In the sublinear case, singular solutions do not exist and all solutions are continuable. The following theorem is a very general form of that result; it is an easy consequence of a result of Hartman [13; Theorem 5.1], so no proof will be given.

Theorem 1. Suppose there exists a continuous non-decreasing function $\omega : R_+ \to R_+$ such that $\int\limits_1^\infty \frac{du}{\omega(u)} = \infty$ and

$$|f(x_1,\ldots,x_n)| \le \omega\left(\sum_{i=1}^n |x_i|\right) \quad \text{on} \quad R_+. \tag{20}$$

Then Eq. (1) does not have any singular solutions and every solution is continuable to all of R_+.

In the case of second order equations, under very mild conditions on the function r, namely, if $r < 0$ is of bounded variation on any finite interval, equation (9) has no singular solutions. Sufficient conditions for the existence of continuable solutions in case (20) does not hold, are given, for example, in [14].

The following theorem gives sufficient conditions for singular solutions of (1) to be of the limit-point type.

Theorem 2. Let $r \neq 0$ be continuous on R_+ and let a function $g \in C^0(R_+)$ exist such that

$$|f(x_1, \ldots, x_n)| \leq g(|x_1|) \quad \text{on } R^n. \tag{21}$$

Then every singular solution y of Eq. (1) is of the limit-point type in the sense of Definition 2.

Proof. Let y be a singular solution of (1) defined on $I = [t_y, b)$, $b < \infty$. Then, using (1), (2), and (21),

$$\begin{aligned} J(t) &\equiv |y^{[n-1]}(t) - y^{[n-1]}(t_y)| = |\int_{t_y}^{t} y^{[n]}(s)\, ds| \\ &\leq \int_{t_y}^{t} |r(s)|\,|f(y^{[0]}, \ldots, y^{[n-1]}(s))|\, ds \leq M \int_{t_y}^{t} g(|y(s)|)\, ds \end{aligned} \tag{22}$$

for $t \in I$, where $M = \max_{t_y \leq s \leq b} |r(s)|$. Since y is a singular solution, $y^{[n-1]}$ is unbounded, and so J is unbounded on I as well. From this and (22) we conclude that

$$y \text{ is unbounded on } I \tag{23}$$

and

$$\int_{t_y}^{b} |f(y^{[0]}(s), \ldots, y^{[n-1]}(s))|\, ds = \infty\,. \tag{24}$$

Let y be nonoscillatory. Then y is monotone in a neighborhood of b (see [1; Lemma 1.4]), and (23) yields the existence of $\tau \in I$ such that $|y(t)| \geq 1$ for $t \in [\tau, b)$. Thus, by (3) and (24),

$$\int_{t_y}^{b} y(t)\, f(y^{[0]}(t), \ldots, y^{[n-1]}(t))\, dt \geq \int_{\tau}^{b} |f(y^{[0]}(t), \ldots, y^{[n-1]}(t))|\, dt = \infty,$$

and so y is of the limit-point type in the sense of Definition 2.

Now let y be oscillatory. Set $N = \{t \in I : |y(t)| \leq 1\}$ and observe that N is the union of intervals (see [1]). Then (3), (21), and (24) yield

$$\begin{aligned}
&\int_{t_y}^{b} y(t)\, f(y^{[0]}(t), \ldots, y^{[n-1]}(t))\, dt \\
&\geq \int_{t_y}^{b} |f(y^{[0]}(t), \ldots, y^{[n-1]}(t))|\, dt - \int_N |f(y^{[0]}(t), \ldots, y^{[n-1]}(t))|\, dt \\
&\geq \int_{t_y}^{b} |f(y^{[0]}(t), \ldots, y^{[n-1]}(t))|\, dt - \int_N g(|y(t)|)\, dt \\
&\geq \int_{t_y}^{b} |f(y^{[0]}(t), \ldots, y^{[n-1]}(t))|\, dt - (b - t_y) \max_{|s| \leq 1} g(s) = \infty\,.
\end{aligned}$$

Thus, y is of the limit-point type in the sense of Definition 2, and this completes the proof of the theorem.

The existence of singular solutions of (1) is studied in [6]. Here we present only one such result of the latter type that is needed in the remainder of this paper. We consider the Emden-Fowler type equation

$$y^{(n)} = r(t)|y|^{\lambda} sgn\, y \tag{25}$$

where $r \in C^0(R_+)$ and $\lambda > 1$.

Theorem 3. Let $a \in R_+$ and let n either be odd or of the form $n = 4k$ for $k \in \{1, 2, \ldots\}$. Suppose there exist positive constants K and K_1 such that

$$-K_1 \leq r(t) \leq -K < 0 \quad \text{for} \quad t \in [a, a+2].$$

Then there exists $M > 0$ such that M does not depend on a and every solution y of (25) satisfying the Cauchy initial conditions

$$y^{(i)}(a) = C_i\,, \quad i = 0, 1, \ldots, n-1, \tag{26}$$

with

$$\sum_{j=0}^{n_0-1} (-1)^{j-1} C_j\, C_{n-j-1} + (-1)^{n_0-1} \frac{n - 2n_0}{2} C_{n_0}^2 > M, \tag{27}$$

and

$$\sum_{j=0}^{n-1} |C_j| > \sqrt{\frac{M}{n_0 + 1}}, \tag{28}$$

where $n_0 = [\![\frac{n}{2}]\!]$, is singular and is defined on $I \subset [a, a+2)$.

The following lemma is needed in order to prove this theorem. Here, we let $L^{\gamma}([a,t])$ denote the space of Lebesgue integrable functions u defined on $[a,t]$ with $||u||_{L^{\gamma}} = \left(\int_a^t |u|^{\gamma}\right)^{1/\gamma}$.

Lemma 1. ([9; Theorem 1]) Let $\alpha, \beta, \gamma \geq 1$ and let $j \in \{0, 1, \ldots, n-1\}$ be such that $\dfrac{n-j}{\alpha} + \dfrac{j}{\beta} \geq \dfrac{n}{\gamma}$. Then there exists $\varrho > 0$ such that

$$||u^{(j)}||_{L^{\gamma}([a,t])} \leq \varrho \left[x^{-j-\frac{1}{\alpha}+\frac{1}{\gamma}} ||u||_{L^{\alpha}([a,t])} + x^{n-j-\frac{1}{\beta}+\frac{1}{\gamma}} ||u^{(n)}||_{L^{\beta}([a,t])}\right]$$

for arbitrary $t > a$, $x \in (0, t-a]$ and $u \in C^n([a,t])$.

Proof of Theorem 3. The proof of this theorem is actually embedded in the proof of Theorem 1 in [7] (set $a = a$, $b = a+2$, $t_0 = a+1$, $\eta_1 = K$, and $\eta_2 = K_1$ there). Let ϱ be given by Lemma 1 with $\alpha = \gamma = \lambda + 1$, $\beta = \frac{\lambda+1}{\lambda}$, and $j = 0, 1, 2, \ldots, n-1$. Note that ϱ does not depend on a as it can be seen from the transformation $\bar{t} = t + \text{const}$. Set

$$\nu = 4\varrho^2 (n - n_0) K_1^{(n-1)/(n - \frac{\lambda-1}{\lambda+1})}, \quad \delta = \sum_{k=0}^{n-1} (k+1)!(2k+1)^{k+1},$$

and

$$\mu = ((\lambda+1)n - \lambda + 1)\left((\lambda+1)n - \lambda + 1 - 2\frac{\lambda-1}{\lambda+1}\right)^{-1} > 1\,.$$

In the proof of Theorem 1 in [7], it is shown that a solution y of Eq. (25) satisfying (26) with

$$\sum_{j=0}^{n_0-1} (-1)^{j-1} C_j C_{n-j-1} + (-1)^{n_0-1} \frac{n-2n_0}{2} C_{n_0}^2$$
$$> K^{-\frac{1}{\mu-1}} \nu^{\frac{\mu}{\mu-1}} (\mu-1)^{-\frac{2}{(\lambda+1)(\mu-1)}}$$

and

$$\sum_{j=0}^{n-1} |C_j| > K^{-\frac{\lambda}{\lambda-1}} (\delta K + K_1)$$

is singular and defined on $I \subset [a, a+2)$. From this it is easy to see that such an M not depending on a does exists.

The following example shows that singular solutions of the limit-circle type do exist.

Example 3. The equation

$$y^{(6)}(t) = \frac{1}{2}\left(\frac{32}{945}\right)^{11} y(t)(y^{(5)}(t))^{12} sgn\, y(t)$$

has the singular solution $y(t) = (b-t)^{9/2}$, $b \in R_+$, that is of the limit-circle type in the sense of Definition 2.

It follows from Theorem 1 that Definition 1 and Definition 2 coincide if (20) holds. Furthermore, it is evident that if Eq. (1) is of the limit-circle type according to Definition 2, then it is also of the limit-circle type according to Definition 1. That the converse relationship does not hold can be seen from the following result.

Theorem 4. Let $\lambda > 1$ and let n either be odd or of the form $n = 4k$ for $k \in \{1, 2, \ldots\}$. Suppose there exist positive constants K and K_1 such that

$$-K_1 \le r(t) \le -K < 0 \text{ for } t \in R_+.$$

Then Eq. (25) is of the limit-circle type in the sense of Definition 1 and it is of the limit-point type in the sense of Definition 2.

Proof. According to Theorem 3, the set of singular solutions of Eq. (25) is nonempty, and Theorem 2 yields that every singular solution is of the limit-point type in the sense of Definition 2. Hence, Eq. (25) is of the limit-point type in the sense of Definition 2.

We now prove that every continuable solution of Eq. (25) is of the limit-circle type. Let y be a continuable solution of (25), set $n_0 = [\![\frac{n}{2}]\!]$, and define

$$F(t) = \sum_{i=0}^{n_0-1} (-1)^{i-1} y^{(n-i-1)}(t) y^{(i)}(t) + (-1)^{n_0-1}\frac{n-2n_0}{2}[y^{(n_0)}(t)]^2 \quad (29)$$

for $t \in R_+$. Then

$$F'(t) = -r(t)|y(t)|^{\lambda+1} - (n-2n_0-1)[y^{(n_0)}(t)]^2 \ge 0, \quad (30)$$

for $t \in R_+$. Thus, F is nondecreasing.

Since the hypotheses of Theorem 3 are satisfied, let M denote the constant given by Theorem 3. We will show that

$$F(t) \le M, \quad t \in R_+ \quad (31)$$

To the contrary, suppose there exists $t_0 \in R_+$ such

$$F(t_0) > M. \tag{32}$$

This implies that (27) holds for $a = t_0$, and so by Theorem 3, (28) does not hold, i.e.,

$$\sum_{j=0}^{n-1} |y^{(j)}(t_0)| \le \sqrt{\frac{M}{n_0+1}}\,.$$

From this and (29), we have

$$F(t_0) \le \frac{M}{n_0+1}(n_0+1) = M\,.$$

This contradicts (32) and proves that (31) holds for $t \in R_+$. Furthermore, (30) and (31) yield

$$\begin{aligned}\int_0^\infty |y(t)|^{\lambda+1}\,dt \;&\le\; \int_0^\infty \frac{F'(t)\,dt}{|r(t)|} \le \frac{1}{K}\int_0^\infty F'(t)\,dt = \frac{1}{K}(F(\infty)-F(0))\\ &\le\; \frac{1}{K}(M-F(0)) < \infty,\end{aligned}$$

and hence y is of the limit-circle type in the sense of Definition 1. Since y was an arbitrary continuable solution of (25), we have that (25) is of the limit-circle type in the sense of Definition 1.

Theorem 4 shows that Definition 1 and Definition 2 are clearly different. Definition 1 is more frequently used since continuable solutions are in general more important in applications than are singular ones. Moreover, Theorem 2 shows that Eq. (1) is of the limit-point type according to Definition 2 nearly always if singular solutions exist.

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The Invadability of a Host in Host-parasitoid Systems*

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Abstract

In this paper, we consider a system of difference equations which describes host-parasitoid interactions in ecosystems. We obtain the criterion for the invadability of a host to the resident host-parasitoid system with a stable periodic orbit. The criterion is useful to investigate the effect of evolution of a host on host-parasitoid systems. Moreover we obtain the necessary condition for the coexistence of an invader host with a resident host and parasitoid. Although two hosts cannot coexist without a parasitoid, there is a possibility that two hosts coexist with the help of a parasitoid.

1 Introduction

The question about the possibility of the coexistence for the interacting species in a community is one of the most important problems in popu-

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lation ecology. The species composing a community can be replaced by the other species, because of the appearance of a mutant species in the community and the invasion of a different species from distant habitats. Therefore, it is necessary not only to investigate the population dynamics of the community composed of fixed member of species, but also to investigate the dynamics involving the effect of evolution. The dynamics of the population model involving the effect of evolution has been investigated (e.g. Godfray *et al.* [5], Metz *et al.* [9], Gatto [4], Hochberg and Holt [6], Abrams and Matsuda [1]).

By analyzing the invadability of the species, Gatto [4] specified the dynamics favored by natural selection. Gatto [4] investigated a single-species population dynamics. But there are many types of interactions between species in natural ecosystem. One of the most popular types is a host-parasitoid interaction. In this paper, we focus on host-parasitoid systems. The purpose of this paper is to obtain the criterion for the invadability of a host to a host-parasitoid system and the condition for the coexistence of the invader host with the resident host and parasitoid. The criterion for the invadability and the condition for the coexistence are useful to investigate the effect of evolution of a host on host-parasitoid systems.

This paper is composed as follows. In Section 2, we give the definition of the invadability and the model used in this paper. In Section 3, we summarize some known results on the invadability of an invader host to a resident host population and on the dynamics after the successful invasion. Further, in Section 4, we investigate the invadability of an invader host to the resident host-parasitoid system and the condition for the coexistence of the invader with the resident host and parasitoid. The final section includes conclusion and contains some future problems.

2 Preliminaries and Model

The model investigated in this paper is described by difference equations, which are used to model the population dynamics with non-overlapping generations such as insects . The dynamics of the system with (resident) k-species is generally given as follows:

$$x_i(n+1) = f_i(x_1(n), \ldots, x_k(n)), \quad i = 1, \ldots, k, \tag{1}$$

where $x_i(n)$ is the population density at generation n of the i-th species for $i = 1, \ldots, k$. To consider the case where an invader species appears in the resident system (1), the following $(k+1)$-dimensional difference equations are used:

$$x_i(n+1) = g_i(x_1(n), \ldots, x_k(n), x_{k+1}(n)) \quad i = 1, \ldots, k+1, \tag{2}$$

where x_{k+1} is the population density of the invader species, $g_i(x_1(n), \ldots, x_k(n), 0) = f_i(x_1(n), \ldots, x_k(n))$ for $i = 1, \ldots, k$ and $g_{k+1}(x_1(n), \ldots, x_k(n), 0) = 0$, that is, system (2) is reduced to system (1) if the invader x_{k+1} is absent. Hereafter a system without an invader (e.g. Eq.(1)) is called a resident system and one with an invader (e.g. Eq.(2)) is called a full system.

In general, the resident system is said to be invadable by an invader if there is a possibility that the density of the invader which is initially rare can increase. But in this paper we confine ourselves to the resident system which has a stable orbit and define the invadability to such a resident system as follows.

Definition 1. Suppose that the resident system has a stable periodic orbit. Let $\{(x_1(i), \ldots, x_k(i))\}_{i=0,\ldots m-1}$ be a stable orbit of period m. The resident system is said to be invadable by an invader if $\{(x_1(i), \ldots, x_k(i), 0)\}_{i=0,\ldots m-1}$ is unstable with respect to $\mathrm{int}\mathbf{R}_+^{k+1}$.

Note that the invadability does not depend on whether the invader survives after the invasion.

The following is a full system investigated in this paper:

$$\begin{cases} H_1(n+1) &= r_1 H_1(n) \exp[-\mu_1(H_1(n) + H_2(n))] \exp[-a_1 P(n)] \\ P(n+1) &= \sum_{i=1}^{2} b_i H_i(n)(1 - \exp[-a_i P(n)]) \\ H_2(n+1) &= r_2 H_2(n) \exp[-\mu_2(H_1(n) + H_2(n))] \exp[-a_2 P(n)], \end{cases} \tag{3}$$

where $H_1(n)$ and $H_2(n)$ are the population densities of two hosts (one is a resident and the other is an invader), $P(n)$ is the density of the resident parasitoid, r_i is the intrinsic rate of increase of H_i, μ_i is the self-regulation parameter of H_i, a_i is the per capita parasitoid attack rate for H_i and b_i is the number of the parasitoid which develops per H_i. The competitive ability of each type of the hosts is identified by the pair

(μ, r, a, b). Model (3), which is an exploitative competitive model, is the special case of the model which was investigated by Comins and Hassell [3]. They considered the effect of a top predator on the stability of the system composed of competing prey species and showed that predators enhance prey species diversity.

As described above, the resident system is system (3) with $H_1(n) = 0$ or $H_2(n) = 0$. Beddington *et al.* [2] carried out a local stability analysis of a positive equilibrium of the resident system, and showed that its dynamics indicates chaotic oscillations depending on the values of the parameters.

Now we introduce the nondimensional quantities by

$$u_1 = \mu_1 H_1,\ u_2 = \mu_1 H_2,\ \ v = a_1 P,\ \alpha = \frac{a_2}{a_1},\ \beta = \frac{\mu_2}{\mu_1},\ c_1 = \frac{b_1}{\mu_1},\ c_2 = \frac{b_2}{\mu_1}.$$

By substituting these quantities, system (3) becomes

$$\left\{\begin{array}{rcl} u_1(n+1) & = & r_1 u_1(n)\exp[-(u_1(n)+u_2(n))]\exp[-v(n)] \\ v(n+1) & = & c_1 u_1(n)(1-\exp[-v(n)]) \\ & & +c_2 u_2(n)(1-\exp[-\alpha v(n)]) \\ u_2(n+1) & = & r_2 u_2(n)\exp[-\beta(u_1(n)+u_2(n))]\exp[-\alpha v(n)]. \end{array}\right. \tag{4}$$

This model contains six positive parameters. In the following section, we investigate the dynamics of this nondimensionalized system. To simplify descriptions, we introduce the following notations : $(4)_{Hoo}$ (or $(4)_{ooH}$) is model (4) with $v(n) = 0$ and $u_2(n) = 0$ (or $u_1(n) = 0$), $(4)_{HoH}$ is one with $v(n) = 0$, $(4)_{HPo}$ (or $(4)_{oPH}$) is one with $u_2(n) = 0$ (or $u_1(n) = 0$).

Before investigating the detailed dynamics of (4), we show that the solution of (4) is bounded. The boundedness is used in the following section.

Theorem 2. The solution of system (4) with $(u_1(0), v(0), u_2(0))^T \in \mathbf{R}^3_+$ is bounded.

Proof. Let $(u_1(n), v(n), u_2(n))^T \in \mathbf{R}^3_+$.
Then $(u_1(n+1), v(n+1), u_2(n+1))^T \in \mathbf{R}^3_+$. We obtain the following inequalities by using the first and third equations of (4):

$$\begin{array}{rcl} u_1(n+1)+u_2(n+1) & \le & r_1 u_1(n)\exp[-(u_1(n)+u_2(n))] \\ & & +r_2 u_2(n)\exp[-\beta(u_1(n)+u_2(n))] \\ & \le & r_M(u_1(n)+u_2(n))\exp[-\beta_m(u_1(n)+u_2(n))], \end{array}$$

where $r_M = \max\{r_1, r_2\}$ and $\beta_m = \min\{1, \beta\}$. Since the function $r_M x \exp[-\beta_m x]$ has a maximum value, $r_M/(\beta_m e)$, we obtain the following inequality:

$$u_1(n+1) + u_2(n+1) \le \frac{r_M}{\beta_m e}.$$

Since $(u_1(n+1), v(n+1), u_2(n+1))^T \in \mathbf{R}^3_+$, we have $u_1(n+1) \le r_M/(\beta_m e)$ and $u_2(n+1) \le r_M/(\beta_m e)$. This shows the boundedness of $u_1(n)$ and $u_2(n)$. We obtain the following inequalities by using the second equation of (4):

$$\begin{aligned} v(n+1) &= c_1 u_1(n)(1-\exp[-v(n)]) + c_2 u_2(1-\exp[-\alpha v(n)]) \\ &\le \sum_{i=1}^{2} c_i u_i(n) \\ &\le 2c_M \frac{r_M}{\beta_m e}, \end{aligned}$$

where $c_M = \max\{c_1, c_2\}$. This shows the boundedness of $v(n)$.

3 Host-Host System

In this section, we describe some known results on the dynamics of system (4) without a parasitoid, that is $(4)_{HoH}$. Because of the absence of the parasitoid, the resident system is model $(4)_{Hoo}$ or $(4)_{ooH}$, and the full system is $(4)_{HoH}$. Systems $(4)_{Hoo}$ and $(4)_{ooH}$ are called the Ricker model. May and Oster [8] showed that its dynamics indicates stable cycles and chaotic oscillations depending on the values of the parameters.

The following theorems are known results on systems $(4)_{Hoo}$, $(4)_{ooH}$ and $(4)_{HoH}$ (see May and Oster [8], Gatto [4], Kocic and Ladas [7]).

Theorem 3. Let $\{u_i(n)\}$ $(i = 1, 2)$ be solutions of system $(4)_{Hoo}$ and $(4)_{ooH}$, respectively. If $\ln r_i > 0$, then $\lim_{n\to\infty} \inf u_i(n) > 0$ for any $u_i(0) > 0$ $(i = 1, 2)$.

From this theorem, we can see that the host u_1 and u_2, when they are isolated from each other, do not go to extinction if $\ln r_1 > 0$ and $\ln r_2 > 0$ respectively.

Theorem 4. Let $\{(u_1(n), u_2(n))\}$ be a solution of system $(4)_{HoH}$ and assume that $\ln r_1 > 0$ and $\ln r_2 > 0$.

(i) If $\ln r_2/(\beta \ln r_1) > 1$, then $\lim_{n\to\infty} u_1(n) = 0$ and $\lim_{n\to\infty} \inf u_2(n) > 0$ for any $(u_1(0), u_2(0))^T \in \mathrm{int}\mathbf{R}^2_+$.

(ii) If $\ln r_2/(\beta \ln r_1) < 1$, then $\lim_{n\to\infty} u_2(n) = 0$ and $\lim_{n\to\infty} \inf u_1(n) > 0$ for any $(u_1(0), u_2(0))^T \in \mathrm{int}\mathbf{R}^2_+$.

Theorem 4 gives the dynamics of system $(4)_{HoH}$ in the case where each host does not go to extinction when two hosts are isolated. In this case, we can see that if the invader host u_2 with the parameters satisfying $\ln r_2/(\beta \ln r_1) > 1$ appears in the resident system, the resident host u_1 goes to extinction and the invader host u_2 can survive. This shows that the resident system is replaced by u_2. Since $\lim_{n\to\infty} \inf u_2(n) > 0$ for any $(u_1(0), u_2(0))^T \in \mathrm{int}\mathbf{R}^2_+$, the condition $\ln r_2/(\beta \ln r_1) > 1$ makes a stable periodic orbit of system $(4)_{Hoo}$ to be unstable with respect to $\mathrm{int}\mathbf{R}^2_+$. According to Definition 1, the condition $\ln r_2/(\beta \ln r_1) > 1$ is the criterion for the invadability to the resident system $(4)_{Hoo}$ with a stable periodic orbit. In Definition 1, we did not define the invadability to the resident system with a non-periodic orbit. But the non-periodic orbit of $(4)_{Hoo}$ is shown to be unstable with respect to $\mathrm{int}\mathbf{R}^2_+$ and $(4)_{HoH}$, if $\ln r_2/(\beta \ln r_1) > 1$. Under this extended sense of the invadability, system $(4)_{Hoo}$ with a non-periodic orbit is also invadable if $\ln r_2/(\beta \ln r_1) > 1$. The invadable set of the parameters in this sense is shown in Fig.1. The resident host u_1 and the invader host u_2 cannot coexist, except for the special case where the parameters satisfy $\ln r_2/(\beta \ln r_1) = 1$. We can see that to invade the resident system without the parasitoid the invader must be superior to the resident host at least in either the ability on the intrinsic rate of increase or on the self-regulation, that is $\ln r_2 > \ln r_1$ or $\beta < 1$. Fig.2 shows an example of the temporal fluctuation of the population density for $(4)_{HoH}$ satisfying $\ln r_1 > 0$, $\ln r_2 > 0$ and $\ln r_2/(\beta \ln r_1) > 1$.

4 Host-Parasitoid-Host System

In this section, we consider the case where the resident system is composed of a host and a parasitoid. Then the resident system is $(4)_{HPo}$ or $(4)_{oPH}$, and the full system is (4).

Numerical investigations show that the resident system $(4)_{HPo}$ and system $(4)_{oPH}$ have stable cycles depending on the parameters (see Beddington *et al.* [2]). The following theorem describes the necessary and sufficient condition for the resident system to have a periodic orbit of period m.

Theorem 5. (i) The system $(4)_{HPo}$ has a periodic orbit of period m if and only if there exists an orbit $\{u_1(i), v(i)\}_{i=0,\ldots,m-1}$ $((u_1(i), v(i)) \neq (u_1(j), v(j)); i, j \in \{0, \ldots m-1\}, i \neq j)$ satisfying the following equations:

$$\ln r_1 = \frac{\sum_{i=0}^{m-1} u_1(i)}{m} + \frac{\sum_{i=0}^{m-1} v(i)}{m} \quad \text{and} \quad v(0) = v(m). \tag{5}$$

(ii) The system $(4)_{oPH}$ has a periodic orbit of period m if and only if there exists an orbit $\{v(i), u_2(i)\}_{i=0,\ldots,m-1}$ $((v(i), u_2(i)) \neq (v(j), u_2(j)); i, j \in \{0, \ldots m-1\}, i \neq j)$ satisfying the following equations:

$$\ln r_2 = \beta\frac{\sum_{i=0}^{m-1} u_2(i)}{m} + \alpha\frac{\sum_{i=0}^{m-1} v(i)}{m} \quad \text{and} \quad v(0) = v(m). \tag{6}$$

Proof. (i) First, let $\{(u_1(i), v(i))\}_{i=0,\cdots,m-1}$ be a periodic orbit of period m of $(4)_{HPo}$. Then we can obtain the following equations by using the first equation of (4).

$$\begin{aligned}
u_1(1) &= r_1 u_1(0)\exp[-u_1(0)]\exp[-v(0)] \\
u_1(2) &= r_1 u_1(1)\exp[-u_1(1)]\exp[-v(1)] \\
&= r_1\{r_1 u_1(0)\exp[-v(0)]\exp[-u_1(0)]\}\exp[-u_1(1)]\exp[-v(1)] \\
&= r_1^2 u_1(0)\prod_{i=0}^{1}\exp[-u_1(i)]\exp[-v(i)] \\
&\vdots \\
u_1(m) &= r_1^m u_1(0)\prod_{i=0}^{m-1}\exp[-u_1(i)]\exp[-v(i)].
\end{aligned} \tag{7}$$

Using $u_1(m) = u_1(0)$ in the above, we obtain (5).

Conversely, from (5) and (7), we have $u_1(m) = u_1(0)$. Then we can see with the assumption $v(m) = v(0)$ that the orbit is periodic with period m.

(ii) It can be proved in similar way as (i).

These conditions (5) and (6) are used to obtain the invadable set of the parameters.

4.1 Invadability

Theorem 6. (i) Let $\{(u_1(i), v(i))\}_{i=0,\dots,m-1}$ be a stable periodic orbit of period m of the resident system $(4)_{HPo}$. If the parameters satisfy the following condition:

$$\ln r_2 > \beta \frac{\sum_{i=0}^{m-1} u_1(i)}{m} + \alpha \frac{\sum_{i=0}^{m-1} v(i)}{m}, \tag{8}$$

then the resident system $(4)_{HPo}$ is invadable by the invader u_2.

(ii) Let $\{(v(i), u_2(i))\}_{i=0,\dots,m-1}$ be a stable periodic orbit of period m of the resident system $(4)_{oPH}$. If the parameters satisfy the following condition:

$$\ln r_1 > \frac{\sum_{i=0}^{m-1} u_2(i)}{m} + \frac{\sum_{i=0}^{m-1} v(i)}{m},$$

then the resident system $(4)_{oPH}$ is invadable by the invader u_1.

Proof. To prove this theorem, we investigate the stability of the periodic orbit $\{\mathbf{p}(i)\}_{i=0,\dots,m-1} = \{(u_1(i), v(i), 0)\}_{i=0,\dots,m-1}$ of system (4). The Jacobian matrix J of (4) evaluated at the periodic orbit $\{\mathbf{p}(i)\}_{i=0,\dots,m-1}$ is given by

$$J = D\mathbf{F}^m(\mathbf{p}(0)) = D\mathbf{F}(\mathbf{p}(m-1)) \cdots D\mathbf{F}(\mathbf{p}(1)) D\mathbf{F}(\mathbf{p}(0))$$

where

$$\begin{aligned}
\mathbf{F}(x,y,z) &= \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} \\
&= \begin{pmatrix} r_1 x \exp[-(x+z)] \exp[-y] \\ c_1 x(1-\exp[-y]) + c_2 z(1-\exp[-\alpha y]) \\ r_2 z \exp[-\beta(x+z)] \exp[-\alpha y] \end{pmatrix} \\
\mathbf{F}^m(\mathbf{p}(i)) &= \mathbf{F}(\mathbf{F}(\cdots \mathbf{F}(\mathbf{p}(i)))) \\
D\mathbf{F}(\mathbf{p}(i)) &= \begin{pmatrix} \left.\frac{\partial F_1}{\partial x}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_1}{\partial y}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_1}{\partial z}\right|_{\mathbf{p}(i)} \\ \left.\frac{\partial F_2}{\partial x}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_2}{\partial y}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_2}{\partial z}\right|_{\mathbf{p}(i)} \\ \left.\frac{\partial F_3}{\partial x}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_3}{\partial y}\right|_{\mathbf{p}(i)} & \left.\frac{\partial F_3}{\partial z}\right|_{\mathbf{p}(i)} \end{pmatrix}.
\end{aligned}$$

Then J is written as

$$J = \begin{pmatrix} \prod_{i=0}^{m-1} A_i & B \\ 0 & \prod_{i=0}^{m-1} C_i \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} \frac{\partial F_1}{\partial x}\Big|_{\mathbf{p}(i)} & \frac{\partial F_1}{\partial y}\Big|_{\mathbf{p}(i)} \\ \frac{\partial F_2}{\partial x}\Big|_{\mathbf{p}(i)} & \frac{\partial F_2}{\partial y}\Big|_{\mathbf{p}(i)} \end{pmatrix}, \quad C_i = \frac{\partial F_3}{\partial z}\Big|_{\mathbf{p}(i)},$$

B is a suitable 2×1 matrix and 0 is a 1×2 zero matrix. Then the stability of the periodic orbit $\{\mathbf{p}(i)\}_{i=0,\ldots,m-1}$ depends only on the value of $\prod_{i=0}^{m-1} C_i$, because the periodic orbit $\{(u_1(i), v(i))\}_{i=0,\ldots,m-1}$ of system $(4)_{HPo}$ is assumed to be stable (the absolute value of eigenvalues of the matrix $\prod_{i=0}^{m-1} A_i$ is smaller than unity). If

$$\left| \prod_{i=0}^{m-1} \frac{\partial F_3}{\partial z}\Big|_{\mathbf{p}(i)} \right| = \prod_{i=0}^{m-1} r_2 \exp[-\beta u_1(i)] \exp[-\alpha v(i)] > 1,$$

that is, if

$$\sum_{i=0}^{m-1} (\ln r_2 - \beta u_1(i) - \alpha v(i)) > 0,$$

the periodic orbit $\{\mathbf{p}(i)\}_{i=0,\ldots,m-1}$ is unstable with respect to u_2-direction and the resident system is invadable by the invader u_2.

(ii) It can be proved in similar way as (i).

The boundary for the invadability to the resident system $(4)_{HPo}$ with a stable periodic orbit of period m is given by (8) with an equality instead of an inequality. The boundary is rewritten as follows by using the necessary condition for the resident system to have a periodic orbit (see Theorem 5 (i)):

$$\frac{1}{\beta} \frac{\ln r_2}{\ln r_1} = \frac{\bar{v}_m}{\bar{u}_m + \bar{v}_m} \frac{\alpha}{\beta} + \frac{\bar{u}_m}{\bar{u}_m + \bar{v}_m}, \tag{9}$$

where $\bar{u}_m = \sum_{i=0}^{m-1} u_1(i)/m$ and $\bar{v}_m = \sum_{i=0}^{m-1} v(i)/m$. The boundary (9), which is the straight line passing through the point $(1, 1)$ with the intercept $\bar{u}_m/(\bar{u}_m + \bar{v}_m)$ and the positive slope $\bar{v}_m/(\bar{u}_m + \bar{v}_m)$ in the $\alpha/\beta - \ln r_2/(\beta \ln r_1)$ parameter space, is shown in Fig.3. From Fig.3, we can see that the superiority of the invader u_2 in the two-host system without a parasitoid (that is, $\ln r_2/(\beta \ln r_1) > 1$, see Theorem 4 (i)) is not sufficient for u_2 to invade the resident system $(4)_{HPo}$.

To obtain the criterion for invadability to the resident system with a non-periodic orbit is a future problem.

4.2 Non-coexistence

Now we examine the condition for the coexistence of three species. From the following theorem, we obtain the necessary condition.

Theorem 7. Let $\{(u_1(n), v(n), u_2(n))\}$ be a solution of system (4).
(i) If $\ln r_2/(\beta \ln r_1) > 1$ and $\alpha/\beta < 1$, then $\lim_{n\to\infty} u_1(n) = 0$ for any $(u_1(0), v(0), u_2(0))^T \in \mathrm{int}\mathbf{R}^3_+$.

(ii) If $\ln r_2/(\beta \ln r_1) < 1$ and $\alpha/\beta > 1$, then $\lim_{n\to\infty} u_2(n) = 0$ for any $(u_1(0), v(0), u_2(0))^T \in \mathrm{int}\mathbf{R}^3_+$.

Proof. To prove that $\lim_{n\to\infty} u_1(n) = 0$ globally in case (i), consider the dynamics of the ratio $u_1(n)/u_2(n)^{1/\beta}$. We can obtain the following equation by using the first and third equations of (4):

$$\begin{aligned}\frac{u_1(n+1)}{u_2(n+1)^{1/\beta}} &= \frac{r_1}{r_2^{1/\beta}} \frac{u_1(n)}{u_2(n)^{1/\beta}} \exp[-(1-\frac{\alpha}{\beta})v(n)] \\ &= \exp[\ln r_1 - \frac{\ln r_2}{\beta}] \exp[-(1-\frac{\alpha}{\beta})v(n)] \frac{u_1(n)}{u_2(n)^{1/\beta}}.\end{aligned}$$

Using the condition on the parameters ($\ln r_2/(\beta \ln r_1) > 1$ and $\alpha/\beta < 1$), we can see that the following equation holds:

$$\lim_{n\to\infty} \frac{u_1(n)}{u_2(n)^{1/\beta}} = 0.$$

By Theorem 2 (the boundedness of $u_2(n)$), we have

$$\lim_{n\to\infty} u_1(n) = 0.$$

We can similarly prove that $\lim_{n\to\infty} u_2(n) = 0$ globally in case (ii).

From this theorem and Theorem 6, we see that if u_2 with the parameters satisfying the condition (i) appears in the resident system $(4)_{HPo}$ with a stable periodic orbit, u_2 can invade and the resident host u_1 goes to extinction.

Theorem 8. Suppose systems $(4)_{HPo}$ and $(4)_{oPH}$ have stable periodic orbits $\{(u_1(i), v_1(i))\}_{i=0,\ldots,m_1-1}$ and $\{(v_2(i), u_2(i))\}_{i=0,\ldots,m_2-1}$ respectively, where m_1 and m_2 are periods of each orbit.

(i) If $\alpha/\beta > 1$ and $\ln r_2/(\alpha \ln r_1) > 1$, then the periodic orbit $\{(u_1(i), v_1(i), 0)\}_{i=0,\ldots,m_1-1}$ is unstable and $\{(0, v_2(i), u_2(i))\}_{i=0,\ldots,m_2-1}$ is stable in system (4).

(ii) If $\alpha/\beta < 1$ and $\ln r_2/(\alpha \ln r_1) < 1$, then the periodic orbit $\{(u_1(i), v_1(i), 0)\}_{i=0,\ldots,m_1-1}$ is stable and $\{(0, v_2(i), u_2(i))\}_{i=0,\ldots,m_2-1}$ is unstable in system (4).

Proof. (i) From the necessary condition for system $(4)_{HPo}$ to have a periodic orbit of period m_1 (see Theorem 5), the following equation holds:

$$\ln r_1 = \frac{\sum_{i=1}^{m_1-1} u_1(i)}{m_1} + \frac{\sum_{i=1}^{m_1-1} v_1(i)}{m_1}. \tag{10}$$

Then

$$\begin{aligned} & \ln r_2 - \beta\frac{\sum_{i=1}^{m_1-1} u_1(i)}{m_1} - \alpha\frac{\sum_{i=1}^{m_1-1} v_1(i)}{m_1} \\ > & \ln r_2 - \alpha(\frac{\sum_{i=1}^{m_1-1} u_1(i)}{m_1} + \frac{\sum_{i=1}^{m_1-1} v_1(i)}{m_1} \\ = & \ln r_2 - \alpha \ln r_1 > 0, \end{aligned} \tag{11}$$

where (10) and the assumptions $\alpha/\beta > 1$ and $\ln r_2/(\alpha \ln r_1) > 1$ are used. From Theorem 6, this shows that $\{(u_1(i), v_1(i), 0)\}_{i=0,\ldots,m_1-1}$ is unstable in system (4).

Now, we investigate the invadability of u_1 to system $(4)_{oPH}$. From the necessary condition for system $(4)_{oPH}$ to have a periodic orbit of period m_2 (see Eq.(6)), the following equation holds:

$$\ln r_2 = \beta\frac{\sum_{i=1}^{m_2-1} u_2(i)}{m_2} + \alpha\frac{\sum_{i=1}^{m_2-1} v_2(i)}{m_2}.$$

Using the above equation, we can obtain the following inequalities:

$$\ln r_1 - \frac{\sum_{i=0}^{m_2-1} v_2(i)}{m_2} - \frac{\sum_{i=0}^{m_2-1} u_2(i)}{m_2}$$

$$= \ln r_1 - \frac{1}{\alpha}(\ln r_2 - \beta\frac{\sum_{i=0}^{m_2-1} u_2(i)}{m_2}) - \frac{\sum_{i=0}^{m_2-1} u_2(i)}{m_2}$$

$$= (\ln r_1 - \frac{\ln r_2}{\alpha}) + (\frac{\beta}{\alpha} - 1)\frac{\sum_{i=0}^{m_2-1} u_2(i)}{m_2} < 0.$$

where the assumptions $\alpha/\beta > 1$ and $\ln r_2/(\alpha \ln r_1) > 1$ are also used. This result shows that $\{(0, v_2(i), u_2(i))\}_{i=0,\dots,m_2-1}$ is stable in system (4).

(ii) It can be proved in similar way as (i).

Although the above theorem gives the local dynamical properties of system (4), it is expected by numerical investigations that the properties are global. Fig.4 shows an example that the resident host goes to extinction and the $(v(n), u_2(n))$ converges to the equilibrium point of system $(4)_{oPH}$. Moreover it is expected by numerical investigations that the following statement is true: if condition (i) in Theorem 8 holds, system $(4)_{HPo}$ has a stable equilibrium (u_1, v_1) with respect to $\text{int}\mathbf{R}^2_+$ and system $(4)_{oPH}$ has a strange attractor, then the orbit with $(u_1(0), v(0), u_2(0)) \in \text{int}\mathbf{R}^3_+$ converges to the strange attractor of $(4)_{oPH}$ (see an example given in Fig.5).

Summarizing the results in the above theorems leads to Fig.6. From Fig.6, we see that the parameters must belong to the region (R3)-i or (R4)-i for three species in system (4) to coexist when systems $(4)_{HPo}$ and $(4)_{oPH}$ have a stable periodic orbit. Fig.7 shows an example of the coexistence of three species. In the case where three species coexist, the replacement of the host does not result from the invasion of a host, but the enhancement of the species diversity results from the invasion. Note that the coexistence of two hosts does not occur without the parasitoid. If the parameters belong to the region (R3)-ii or (R4)-ii in Fig.6, there is a possibility that system (4) is bistable (see Fig.8). The bistability means that two hosts are mutually non-invadable. That is, the system with the host which coexists with the parasitoid is not invadable by the other host.

5 Conclusion

We have investigated the invadability of the host in the resident host-parasitoid system with a stable periodic orbit [illegible] and obtained the criterion for the invadability. The criterion for the invadability obtained in this paper can be applied only when the resident system has a stable periodic orbit. To obtain the criterion for the resident system containing a non-periodic orbit is a future problem.

In Theorem 7, the sufficient conditions for the non-coexistence of three species (two hosts and one parasitoid) are obtained. Theorem 8 and numerical investigations also give the sufficient condition for non-coexistence of three species in the case where system $(4)_{HPo}$ and $(4)_{oPH}$ have stable periodic orbits. We see that if the parameters do not satisfy these conditions, there is a possibility that two hosts coexist with the help of a parasitoid. Note that the two host cannot coexist without a parasitoid (Gatto [4], Kocic and Ladas [7]).

We cannot know from these theorems whether the remaining host and parasitoid coexist or not, after the resident host goes to extinction. But numerical investigations suggest that if the invader host and the parasitoid coexist in system $(4)_{oPH}$, they also coexist after the resident host goes to extinction.

By using the results obtained in this paper, we can investigate the effect of evolution of the host on host-parasitoid systems as follows. First, we check if the resident host-parasitoid system has a stable positive equilibrium. Second, we obtain the host which can invade the resident system by using the criterion for the invadability. Third, if the parameters satisfy the sufficient condition for the non-coexistence of three species, which was obtained in this paper, we see that the resident host goes to extinction. Hence, the effect of evolution of host on host-parasitoid system is examined by investigating the dynamics of the system composed of the invader host and the parasitoid. If three species coexist, we should further investigate the invadability of a host to the resident system composed of two hosts and a parasitoid. It is a future problem.

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Figure legends:

Figure 1: The $\beta - \frac{\ln r_2}{\ln r_1}$ parameter space. Here $\ln r_1 > 0$ and $\ln r_2 > 0$. The straight line is the boundary of the invadability, $\frac{\ln r_2}{\ln r_1}=\beta.$ The hatched region is the invadable set of the parameters. If the parameters belong to

the invadable set, the resident system $(4)_{Hoo}$ is invadable by the invader u_2 and $\lim_{n\to+\infty} u_1(n) = 0$. if the parameters belong to the region which is not hatched, the resident system $(4)_{Hoo}$ is not invadable by the invader u_2 and $\lim_{n\to+\infty} u_2(n) = 0$.

Figure 2: An example of the temporal fluctuations of the population density of system $(4)_{HoH}$ with the parameters belonging to hatched region in Fig.1. The resident system $(4)_{Hoo}$ and system $(4)_{HoH}$ have a stable equilibrium, respectively. The parameters are $\ln r_1 = 1, \ln r_2 = 1.23$ and $\beta = 1$. The initial condition is $(u_1(0), u_2(0)) = (1, 0.000001)$. After the invasion of $u_2, \lim_{n\to+\infty} u_1(n) = 0$ and $u_2(n)$ converges to the equilibrium of system $(4)_{ooH}$.

Figure 3: The $\frac{\alpha}{\beta} - \frac{\ln r_2}{(\beta \ln r_1)}$ parameter space. The straight line is the boundary of the invadability to the resident system $(4)_{HPo}$ with a stable periodic orbit of period m (Eq.(9)). The hatched region is the invadable set of the parameters. If the parameters belong to the invadable set, the resident system $(4)_{HPo}$ is invadable by u_2. If not, $(4)_{HPo}$ is not invadable by u_2.

Figure 4: An example of the temporal fluctuations of the population density of system (4). The resident system $(4)_{HPo}$ has a stable equilibrium $(u_1, v_1) = (0.605183, 0.394817)$. System $(4)_{oPH}$ also has a stable equilibrium $(u_2, v_2) = (0.563833, 0.573400)$. The parameters satisfy $\frac{\alpha}{\beta} > 1$ and $\frac{\ln r_2}{(\alpha \ln r_1)} > 1$. After the invasion of $u_2, \lim_{n\to+\infty} u_1(n) = 0$ and $(v(n), u_2(n))$ converges to the equilibrium of system $(4)_{oPH}$. The parameters are $\ln r_1 = 1, \ln r_2 = 1.25, \beta = 1, \alpha = 1.2, c_1 = 2$ and $c_2 = 2$. The initial condition is $(u_1(0), v(0), u_2(0)) = (0.605183, 0.394817, 0.000001)$.

Figure 5: An example of the temporal fluctuations of the population density of system (4). The resident system $(4)_{HPo}$ has a stable equilibrium $(u_1, v_1) = (0.605183, 0.394817)$. System $(4)_{oPH}$ has a strange attractor. The parameters satisfy $\frac{\alpha}{\beta} > 1$ and $\frac{\ln r_2}{(\alpha \ln r_1)} > 1$. After the invasion of $u_2, \lim_{n\to+\infty} u_1(n) = 0$ and
$(v(n), u_2(n))$ tends to the strange attractor of system $(4)_{oPH}$. The parameters are $\ln r_1 = 1, \ln r_2 = 2.1, \beta = 1, \alpha = 1.5, c_1 = 2$ and $c_2 = 2$. The initial condition is
$(u_1(0), v(0), u_2(0)) = (0.605183, 0.394817, 0.000001)$.

Figure 6: The $\frac{\alpha}{\beta} - \frac{\ln r_2}{(\beta \ln r_1)}$ parameter space. The straight line is the boundary of the invadability to the resident system $(4)_{HPo}$ with a periodic orbit of period m (Eq.(9)). The sufficient condition for the non-coexistence of three species given in Theorem 7, $\{\frac{\ln r_2}{(\beta \ln r_1)} > 1$ and $\frac{\alpha}{\beta} < 1\}$ or $\{\frac{\ln r_2}{(\beta \ln r_1)} < 1$ and $\frac{\alpha}{\beta} > 1\}$ corresponds to the regions (R_1) and (R_6) respectively. The sufficient condition for the non-coexistence of three species given in Theorem 8, $\{\frac{\alpha}{\beta} > 1$ and $\frac{\ln r_2}{(\alpha \ln r_1)} > 1\}$ or $\{\frac{\alpha}{\beta} < 1$ and $\frac{\ln r_2}{(\alpha \ln r_1)} < 1\}$ corresponds to the regions (R_2) and (R_5) respectively. Hence the coexistence of three species is possible only for the parameters belonging to the regions $(R_3) - i$ and $(R_4) - i$. By numerical investigations we see that three species coexist if the parameters belong to the hatched region in the regions $(R_3) - i$ and $(R_4) - i$. The coexistence is determined by the criterion that all species have average population densities during 100 iterations after 10^4 initial iterations greater than $10^{-}4$ when the initial condition is $(u_1(0), v(0), u_2(0)) = (0.1, 0.1, 0.1)$. The parameters are $\ln r_1 = 1, \beta = 1, c_1 = 2$ and $c_2 = 2$. The resident system $(4)_{HPo}$ has an equilibrium $(0.605183, 0.394817)$.

Figure 7: An example of the temporal fluctuations of the population density of system (4) with the parameters belonging to the hatched region in Fig.6. Three species coexist with a stable equilibrium. The parameters are $\ln r_1 = 1, \ln r_2 = 1.23 \beta = 1, \alpha = 1.5, c_1 = 2$ and $c_2 = 2$. The initial condition is $(u_1(0), v(0), u_2(0)) = (0.605183, 0.394817, 0.000001)$.

Figure 8: The temporal fluctuations in the phase space (u_1, v, u_2) for (4) with the parameter belonging to $(R_4) - ii$. The orbits $\{u_1(i), v(i), u_2(i)\}_{i=0,\cdots,1000}$ with the initial condition $(u_1(0), v(0), u_2(0)) = (1, 1, 0.01)$ and $\{u_1(i), v(i), u_2(i)\}_{i=1000,\cdots,2000}$ with the initial condition $(u_1(0), v(0), u_2(0)) = (0.01, 1, 1)$ are shown. The former, of which points are connected with lines, converges to a positive equilibrium of system $(4)_{HPo}$ and the latter tends to an attractor of system $(4)_o PH$. The system (4) is bistable. The parameters are $\ln r_1 = 1, \ln r_2 = 1.35 \beta = 1, \alpha = 1.89, c_1 = 2$ and $c_2 = 2$.

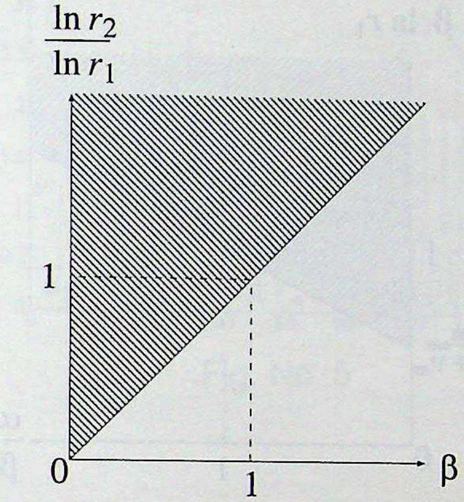
$\frac{\ln r_2}{\ln r_1}$
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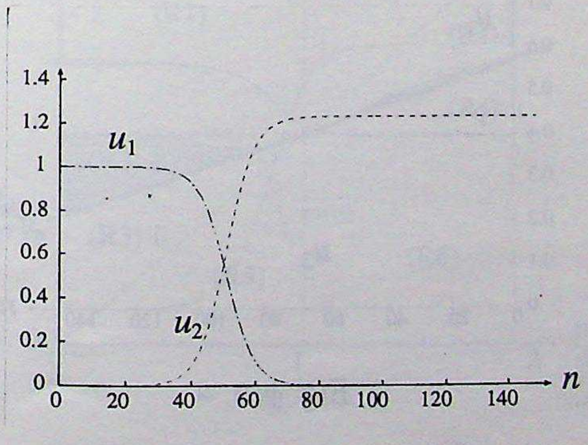
Fig. No. 1



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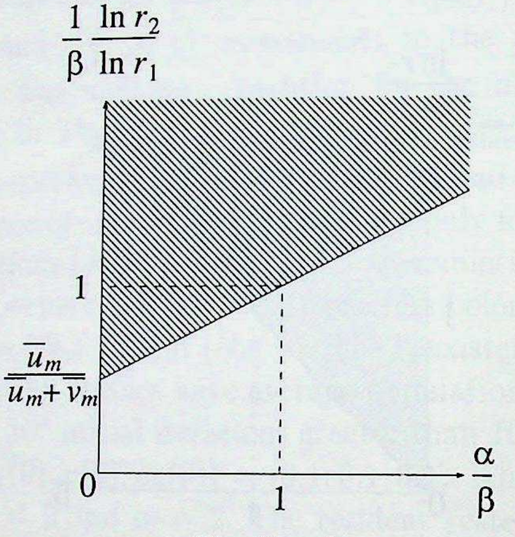

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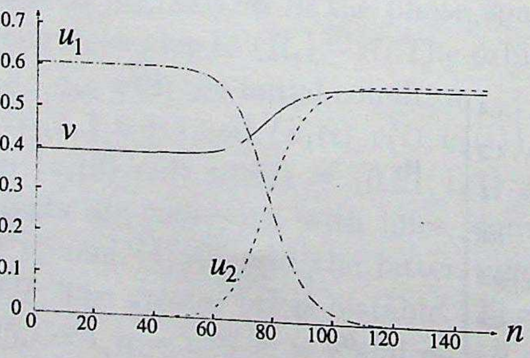

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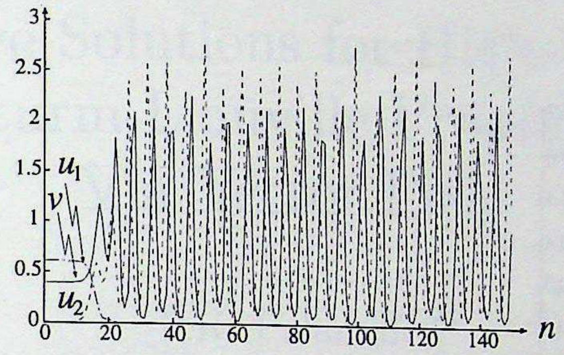

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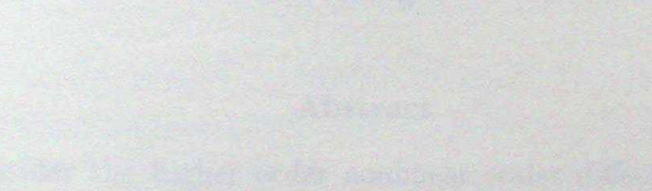

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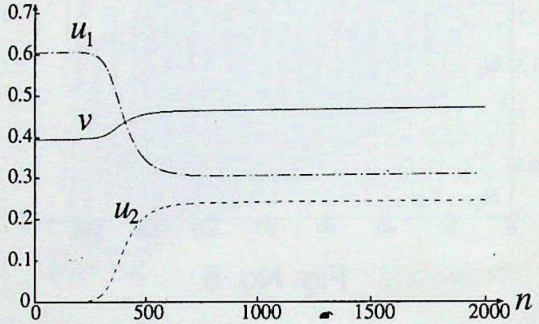

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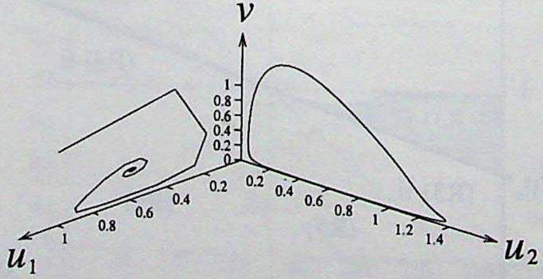

Fig. No. 8

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Positive Solutions for Higher Order Sturm-Liouville Problems Via Vector Field

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Abstract

Consider the higher order nonlinear scalar differential equations

$$x^{(2n)}(t) = -f(t, x(t)), \quad 0 \leq t \leq 1 \tag{0.1}$$

where $f \in C([0,1] \times [0,\infty), (0,\infty))$, associated to the boundary conditions

$$\left\{ \begin{array}{l} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0 \\ \gamma_i x^{(2i)}(1) \pm \delta_i x^{(2i+1)}(1) = 0, \end{array} \right. \tag{0.2}$$

with $\alpha_i, \gamma_i, \delta_i \geq 0$, $\beta_i > 0$ $(i = 0, 1, ..., n-1)$. Existence of a solution of above boundary value Problems (BVP^s) such that

$$x^{(2i)}(t) > 0, \; 0 \leq t \leq 1, \; i = 0, 1, ..., n-1$$

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are given, under superlinear and/or sublinear growth in f. The approach is based on an analysis of the corresponding vector field on the face-plane and the well-known from combinatorial analysis, Knaster-Kuratowski-Mazurkiewicz's principle or as it is known, Sperner's Lemma.

1 Introduction

In [10], Erbe and Tang observed that, if the partial differential equation

$$-\Delta u = F(|x|, u) \text{ in } R < |x| < \hat{R}, \tag{1.1}$$

subject to certain conditions on $|x| = R$ and $|x| = \hat{R}$, is radially symmetric, then it can be transformed to a scalar Sturm-Liouville one

$$x''(t) = -f(t, x(t)), \quad 0 \le t \le 1, \tag{1.2}$$

$$\begin{cases} \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0. \end{cases} \tag{1.3}$$

where the constants $\alpha, \beta, \gamma, \delta \ge 0$. This of course does not mean that any (nonradial) solution of (1.1) is also a solution of (1.2) but rather that existence results for the latter equation guarantee similar conclusions for (1.1).

The literature for the last BVP is voluminous. Suggestively we refer to [2], [9], [13] [16], [18] and the references therein.

In [11], Erbe and Wang by using Green's functions and Krasnoselskii's fixed point theorem in cones, proved existence of a positive solution of (1.2)-(1.3), under the following assumptions:

$(A.1)$ f is continuous and positive, i.e. $f \in C([0,1] \times [0,\infty), (0,\infty))$,

$$(A.2^*) \qquad \begin{cases} f_0 := \lim_{x\to 0+} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0, \\ f_\infty := \lim_{x\to +\infty} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty \end{cases}$$

i.e. f is *supelinear* at both ends points $x = 0$ and $x = \infty$ or

$$(A.2_*) \qquad \begin{cases} f_0 := \lim_{x\to 0+} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty, \\ f_\infty := \lim_{x\to +\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0. \end{cases}$$

i.e. f is *sublinear* at both $x = 0$ and $x = \infty$ and $\rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0$. Erbe and Tang in [10] for $n = 1$ and Davis, Erbe and Henderson in [6] for arbitrary $n = 1, 2, \dots$, established criteria for the existence of multiple positive solutions of

$$(-1)^n x^{(2n)}(t) = f(t, x(t)), \quad 0 \le t \le 1, \tag{1.4}$$

$$\begin{cases} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \\ \gamma_i x^{(2i)}(1) + \delta_i x^{(2i+1)}(1) = 0 \end{cases} \tag{1.5}$$

where

$$(A.3) \qquad \rho_i := \beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0, \ 0 \le i \le n-1,$$

under certain growth rate assumptions on f, for the case when

$$\alpha_i = \alpha, \ \beta_i = \beta, \ \gamma_i = \gamma \ \text{ and } \ \delta_i = \delta, \ 0 \le i \le n-1.$$

It is the aim of this work to prove existence of solutions for the more general boundary value problem (1.6)-(1.5), where

$$x^{(2n)}(t) = -f(t, x(t)), \quad 0 \le t \le 1 \tag{1.6}$$

such that

$$x^{(2i)}(t) > 0, \ 0 \le t \le 1, \ i = 0, 1, \dots, n-1.$$

Further we prove existence of a solution $x = x(t)$ of (1.6)-(1.7) such, that

$$x^{(2i)}(t) > 0, \ x^{(2i+1)}(t) > 0, \ 0 \le t \le 1, \ i = 0, 1, \dots, n-1.$$

where

$$\begin{cases} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0 \\ \gamma_i x^{(2i)}(1) - \delta_i x^{(2i+1)}(1) = 0 \end{cases} \tag{1.7}$$

under the assumption

$$(A.3^*) \qquad \rho_i^* := \beta_i\gamma_i + \alpha_i\gamma_i - \alpha_i\delta_i < 0, \ 0 \le i \le n-1,$$

instead of $(A.3)$ and this is a new result.

Either the nature of equation (1.6) or else the boundary conditions (1.5) or (1.7) possibly make Erbe and Wang's method not applicable, since we do not have nonnegative Green's functions for the homogeneous problems

$$x^{(2n)}(t) = 0,\ n = 1, 2, \dots$$

satisfying the boundary conditions (1.5) or (1.7).

In [6], some growth conditions on

$$h(t,x) := \frac{f(t,x)}{x},\ 0 < x < \infty,\ 0 \le t \le 1$$

established ($h_0(t) = \lim_{x \to 0+} \frac{f(t,x)}{x} = 0$ corresponds to superlinear case while $h_\infty(t) = \lim_{x \to \infty} \frac{f(t,x)}{x} = \infty$ to sublinear one) to yields existence and multiplicity criteria for (1.4)-(1.5). Although we could discuss such a general case, we restrict our consideration only for the case of simpler assumptions ($A.2^*$) or ($A.2_*$) and so we examine here only existence results.

In [19], we noticed that the differential equation (1.6) defines a vector field (see Remark 1.3 below), the properties of which combined with the Kneser's property (continuum) of the cross-sections of the solutions funnel yields existence results for the case $n = 1$. However the technique presented there, does not work anymore for $n > 1$, although cross sections are still continua. So we will now apply another well-known result, namely the Knaster-Kuratowski-Mazurkiewicz's principle (Sperner's Lemma) instead of the Kneser's continuum.

Finally in a recent paper [20], the author studied the $(n_1, n_2, \dots n_k)$ conjugate boundary value problem

$$\begin{cases} x^{(n)}(t) = f(t, x(t)), & 0 \le t \le 1, \\ x^{(j)}(a_i) = 0, & 0 \le j \le n_i - 1,\ 1 \le i \le k, \end{cases}$$

where $n \ge 2$, $n_i \ge 1$ and $2 \le k \le n$ are integers with $\sum_{i=1}^{k} n_i = n$ and $0 = a_1 < a_2 < \dots < a_k = 1$. Specifically existence of a positive solution of this BVP was proved, also by means of Sperner's Lemma.

It will be convenient to represent the differential equation (1.6) as a second order system of the form

$$X''(t) = F(t, X) \tag{1.8}$$

where for notational purpose, we set

$$\begin{cases} X = (x_0, x_1, ..., x_{n-1}) = (x, x'', ..., x^{(2n-2)}) \in \mathbf{R}^n \text{ and} \\ F(t, X) = (x'', ..., x^{(2n-2)}, -f(t, x)). \end{cases}$$

If further we set

$$\zeta = (\zeta_0, \zeta_1, ..., \zeta_{n-1}) \text{ where } \zeta_i = \alpha_i, \beta_i, \gamma_i \text{ or } \delta_i,$$

then the boundary conditions (1.5) take the form:

$$\alpha X(0) - \beta X'(0) = 0 \text{ and } \gamma X(1) + \delta X'(1) = 0, \tag{1.9}$$

where

$$\begin{aligned} \alpha X(0) - \beta X'(0) &:= (\alpha_0 x(0) - \beta_0 x'(0), \cdots, \\ &\qquad \alpha_{n-1} x^{(2n-2)}(0) - \beta_{n-1} x^{(2n-1)}(0)), \\ \gamma X(1) + \delta X'(1) &:= (\gamma_0 x(1) + \delta_0 x'(1), \cdots, \\ &\qquad \gamma_{n-1} x^{(2n-2)}(1) + \delta_{n-1} x^{(2n-1)}(1)). \end{aligned}$$

Solutions of (1.8) are defined by trajectories with their initial values in

$$E_{i0} := \{(X, X') \in \mathbf{R}^{2n} : \alpha_i x_i - \beta_i x_i' = 0,\ x_i \geq 0,\ x_j = 0 = x_j',\ j \neq i\},$$

i.e. if they satisfy

$$\begin{cases} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \\ x^{(2j)}(0) = 0 = x^{(2j+1)}(0),\ 0 \leq j \leq n-1,\ j \neq i. \end{cases} \tag{1.10}$$

A solution of the initial value problem (1.8)-(1.10) will be denoted by

$$X = (X(t, P), X'(t, P)) \text{ or simply } x = x(t; P).$$

where of course $P \in E_{i0}$ and we will represent that fact by $x \in \mathcal{X}(E_{i0})$. Let us denote by

$$\mathbf{K} := \{(X, X') : X \geq 0 \text{ and } \gamma X + \delta X' \geq \gamma X(1) + \delta X'(1) = 0\}.$$

the cone of $\mathbf{R}^{2n}$ (inequalities are considered component-wise) and let $\partial\mathbf{K}$ be its boundary, which consists of the planes

$$E_{i1} := \{(X, X') : \gamma_i x_i + \delta_i x_i' = 0,\ x_i \geq 0,\ x_j = 0 = x_j',\ j \neq i\}.$$

Then it is clear that $E_{i0} \subseteq \mathbf{K}$, $(i = 0, 1, \ldots, n-1)$.

Definition 1.1. We will say that the trajectory $X(t, P)$, $P \in E_{i0}$, egresses from $\mathbf{K}$ through E_{i1} (at the time t_1), whenever there exists a $t_1 > 0$ and $\varepsilon > 0$ such that

$$\begin{cases} \gamma_i x_i(t) + \delta_i x_i'(t) > 0\,, & 0 \leq t < t_1, \\ \gamma_i x_i(t_1) + \delta_i x_i'(t_1) = 0 \text{ and} & \\ \gamma_i x_i(t) + \delta_i x_i'(t) < 0, & t_1 < t < t_1 + \varepsilon \end{cases}$$

and

$$\begin{aligned} \gamma_j x_j(t) + \delta_j x_j'(t) \quad &> 0,\ \ 0 < t < t_1 + \varepsilon,\ j < i \ \text{ and} \\ \gamma_j x_j(t) + \delta_j x_j'(t) \quad &< 0,\ \ 0 < t < t_1 + \varepsilon,\ j > i. \end{aligned}$$

Remark 1.2. By above definition, it is obvious that a trajectory $X(t, P)$, $P \in E_{i0}$, does not "egress" (in our sense) from $\mathbf{K}$ through E_{i1} at the time t_1, if it has been already egressed through another face E_{j1} at a previous time $t_2 \in [0, t_1)$.

Remark 1.3. We notice here that the differential equation (1.6) (or (1.8)) defines a vector field, the properties of which will be crucial for our study. Specifically and recalling the notations

$$x_i = x^{(2i)}(t) \ \text{ and } \ x_i' = x^{(2i+1)}(t)\ ,\ \ 0 \leq i \leq n-1,$$

let us look at the (x_i, x_i') face semi-plane $(x_i \geq 0)$. By the sign condition on f (see assumption (A.1)) we immediately see that for any $x \in \mathcal{X}(E_{i0})$ it follows that $x^{(2n)}(t) < 0$ and then

$$x^{(j)}(t) < 0,\ 2i + 2 \leq j \leq 2n$$

for all $t \in (0, t_1)$ and some $t_1 > 0$. Thus any "trajectory" $(x^{(2i)}(t), x^{(2i+1)}(t))$, $t \geq 0$, emanating from the "semi-line" E_{i0} "trends" in a natural way, initially (when $x^{(2i+1)}(t) > 0$) toward the positive x_i

"semi-axis" and then (when $x^{(2i+1)}(t) < 0$) turns toward the "semi-line" E_{i1}.

These properties will be referred to as *the nature of the vector field* throughout the rest of paper.

Lastly, by setting a certain growth rate on f (say superlinearity) we can control the vector field, so that any solution $x \in \mathcal{X}(P)$, with large enough $P \in E_{i0}$ semi egress strictly from $\mathbf{K}$ through E_{i1}, at some time $t_1 \leq 1$ and for small enough $P \in E_{i0}$, x can not egress from $\mathbf{K}$ through E_{i1}, for any $t \leq 1$.

So the technique presented here is different to that given in the above mentioned papers. Actually, we relay on the above " nature of the vector field" and the Sperner's property, which is formulated in the sequel.

Let $p_0, p_1, ..., p_m$ be $m+1$ points of the m-dimensional Euclidean space $\mathbf{R}^m$. Then the *simplex* $S = [p_0 p_1 ... p_m]$ is defined by

$$S := \Big\{ p \in \mathbf{R}^{2m} : \neq \lambda_i > 0 \ \text{ with } \ \sum_{i=0}^{m} \lambda_i = 1 \ \text{ and } p = \sum_{i=0}^{m} \lambda_i p_i \Big\}.$$

The points $p_0, p_1, ..., p_m$ are called *vertices* of the simplex and the simplex $[p_{i_0} p_{i_1} ... p_{i_k}]$, $0 \leq k \leq m-1$, is a *face* of S. If the vectors $p_0, p_1, ..., p_m$ are linearly independent, then S is an m-dimensional simplex spanned by these points.

Our principle is based on the following result from combinatorial analysis, known as Sperner's lemma ([1] or [14] Ch. II, Th. 5, p. 310).

Lemma 1.4. Let T^m be a closed m-simplex with vertices $\{e^0, e^1, ..., e^m\}$ and let $\{E_1, E_2, ... \ ..., E_m\}$ be a closed covering of T^m such that each closed face $[e^{i_0} e^{i_1} ... e^{i_k}]$ of T^m is containing in the union $E_{i_0} \cup E_{i_1} \cup ... \cup E_{i_k}$. Then the intersection $\cap_{i=0}^{m} E_i$ is nonempty.

2 Main Results

The next lemma shows that, if a trajectory satisfies a certain initial condition of type (1.10), then it egresses from $\mathbf{K}$ at a time $t_1 \leq 1$, through the hyperplanes E_{i1}, for any $i = 0, 1, ..., n-1$.

Lemma 2.1. Assume (A.1), (A.3) and (A.2*) (i.e. the function f is superlinear) hold. Then for each $i = 0, 1, .., n-1$

- there exists an $H_i > 0$ such that for any $P_{i1} := (0, 0, ...0, x_1, x_1', 0, ..., 0) \in E_{i0}$ with $x_1 \geq H_i$ every trajectory $X(t, P_{i1})$ egresses from $\mathbf{K}$ through E_{i1} at a time $t_1 \leq 1$.
- there exist an $\eta_i > 0$ and $P_{i0} := (0, 0, ...0, x_0, x_0', 0, ..., 0) \in E_{i0}$ with $x_0 \leq \eta_i$ such that trajectories $X(t, P_{i0})$ do not egress from $\mathbf{K}$ through E_{i1} (stay "asymptotic" in $\mathbf{K}$) for all time $t \in [0, 1]$.

Proof. Since $f_\infty = +\infty$, for any

$$K > \max\left\{(2n)!\left[\frac{\beta_i(2i+1)+\alpha_i}{\beta_i(2i+1)!}\right], \frac{(2n-2i)!\rho_i}{(\gamma_i + 2\delta_i(n-i))\beta_i}\right\}$$

there exists $H > 0$ such that

$$\min_{0 \leq t \leq 1} f(t, x) > Kx, \quad x \geq H. \tag{2.1}$$

Recalling the notation

$$X := (x, x'', ..., x^{(2n-2)}) \in \mathbf{R}^n,$$

let $X(t, P)$ be a solution satisfying the initial conditions (1.10), i.e.

$$\begin{cases} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \\ x^{(j)}(0) = 0, \ j \neq i. \end{cases} \tag{2.2}$$

Then of course we have

$$x^{(2i+1)}(0) = \frac{\alpha_i}{\beta_i} x^{(2i)}(0) > 0. \tag{2.3}$$

Therefore, by the assumption (A.1) (the nature of the vector field, see Remark 1.3) there exists a $\bar{t} > 0$ such that for $0 < t < \bar{t}$,

$$\begin{cases} x^{(j)}(t) > 0, \ \text{ for } \ j = 0, 1, ..., 2i+1 \text{ and} \\ x^{(j)}(t) < 0, \ \text{ for } \ j = 2i+2, ..., 2n. \end{cases} \tag{2.4}$$

We set

$$x^{(2i)}(0) := x_1 \quad \text{and} \quad x^{(2i+1)}(0) := x_1', \tag{2.5}$$

consider any point $P := (0, 0, ...0, x_1, x'_1, 0, ..., 0) \in E_{i0}$ with $x_1 \geq H$ and let $x \in \mathcal{X}(P)$ be any solution of (1.6) starting at the point P.

Considering the map

$$G_k(t; P) := \gamma_k x^{(2k)}(t) + \delta_k x^{(2k+1)}(t),\ 0 \leq t \leq 1$$

and since by (2.3)

$$G_i(0; P) = \frac{\alpha_i \delta_i + \beta_i \delta_i}{\beta_i} x^{(2i)}(0) > 0,$$

it is obvious by (2.4), that the first zero of $G_k(t; P)$ must occur for $k = i$, i.e. if some solution $x \in \mathcal{X}(P)$ egresses from **K**, it must egress through the planes E_{i1}, $(i = 0, 1, ..., n-1)$.

We are going to prove that the solution x egresses from **K**, provided that x_1 is sufficiently large.

- Let us suppose that for every $H_i \geq H$, there is a point $P \in E_{i0}$ with $x_1 \geq H_i$, such that

$$G_i(t; P) > 0,\ 0 \leq t \leq 1. \tag{2.6}$$

We fix such a point P and for simplicity set $x(t) := x(t; P)$. Now by the Taylor's formula

$$x(1) = \sum_{j=0}^{2n-1} \frac{1}{j!} x^{(j)}(0) + \frac{1}{(2n)!} x^{(2n)}(\hat{t}),$$

and the definition of E_{i0}, we have, for some $\hat{t}, t^* \in [0, 1]$

$$\begin{cases} x^{(2i)}(1) = x_1 + x'_1 - \frac{1}{(2n-2i)!} f(\hat{t}, x(\hat{t})), \\ x^{(2i+1)}(1) = x'_1 - \frac{1}{(2n-2i-1)!} f(t^*, x(t^*)). \end{cases}$$

Also since $P \in E_{i0}$, we get

$$\begin{cases} x^{(2i)}(1) = x_1[1 + \frac{\alpha_i}{\beta_i}] - \frac{1}{(2n-2i)!} f(\hat{t}, x(\hat{t})) \ \text{ and} \\ x^{(2i+1)}(1) = x_1 \frac{\alpha_i}{\beta_i} - \frac{1}{(2n-2i-1)!} f(t^*, x(t^*)) \end{cases}$$

and then ($G_i(P) := G_i(1,P)$)

$$G_i(P) = x_1\left[\frac{\rho_i}{\beta_i} - \frac{\gamma_i}{(2n-2i)!}\frac{f(\hat{t},x(\hat{t}))}{x_1} - \frac{\delta_i}{(2n-2i-1)!}\frac{f(t^*,x(t^*))}{x_1}\right],$$

where we recall that $\rho_i := \beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$.

Assume first that

$$x_1 \leq \min\{x(\hat{t}), x(t^*)\}. \tag{2.7}$$

Thus in view of (2.1) and since $x_1 \geq H_i \geq H$, we get

$$\begin{aligned} G_i(P) \leq \quad & x_1\left[\tfrac{\rho_i}{\beta_i} - \tfrac{\gamma_i}{(2n-2i)!}\tfrac{Kx(\hat{t})}{x_1} - \tfrac{\delta_i}{(2n-2i-1)!}\tfrac{Kx(t^*)}{x_1}\right] \\ & \leq x_1\left[\tfrac{\rho_i}{\beta_i} - \tfrac{(\gamma_i+2\delta_i(n-i))K}{(2n-2i)!}\right]. \end{aligned}$$

So by the choice $K > \frac{(2n-2i)!\rho_i}{(\gamma_i+2\delta_i(n-i))\beta_i}$, we conclude that

$$G_i(P) < 0,$$

contrary to (2.6).

Therefore we may assume that there exist sequences $\{P_n\} \subseteq E_{i0}$, $P_n := (0,...,0,x_{1n},x'_{1n},0,...,0)$, with $\lim x_{1n} = +\infty$, $x_n \in \mathcal{X}(P_n)$ and $\left\{\hat{t}_n\right\}$ such that

$$x_n(\hat{t}_n) < x_1, \ n = 1,2,... \tag{2.8}$$

Then by the Taylor's formula and the monotonicity of $x_n(t)$ (which follows by (2.6)), we get for some $\bar{t}_n \in (0,\hat{t}_n)$,

$$\left\{\begin{aligned} x_n(\hat{t}_n) \quad & = x_{1n}\tfrac{\hat{t}_n^{2i}}{(2i)!}[1 + \tfrac{\alpha_i}{(2i+1)\beta_i}\hat{t}_n] - \tfrac{1}{(2n)!}f(\bar{t}_n, x(\bar{t}_n)) \\ & \geq x_{1n}\tfrac{\hat{t}_n^{2i}}{(2i)!}[1 + \tfrac{\alpha_i}{(2i+1)\beta_i}\hat{t}_n] - M, \end{aligned}\right. \tag{2.9}$$

where $M := \frac{1}{(2n)!}\max\{f(t,x) : 0 \leq t \leq 1, 0 \leq x \leq x_1\}$. Obviously 0 cannot be an accumulation point of $\left\{\hat{t}_n\right\}$, since at the opposite case (up to a subsequence) we get

$$\lim x_n(\hat{t}_n) = \lim x_{1n} = +\infty,$$

a contradiction to (2.8). Consequently it is clear that (2.9) also contradicts (2.8). Hence (2.6) cannot be true and the first result of Lemma follows.

Similarly by the *superlinearity* of $f(t,x)$ at $x=0$, for any $\mu>0$ there is an $\eta>0$ such that

$$0 < x \leq \eta \quad \text{implies} \quad \max_{0\leq t\leq 1} f(t,x) < \mu x. \tag{2.10}$$

Consider any positive number $\varepsilon < \min\{1, A_i^{-1}\}$ and choose

$$\mu < \min\left\{\frac{\varepsilon \rho_i (2n-2i)!}{\beta_i[\gamma_i + 2(n-i)\delta_i]},\ (2n)!(1-\varepsilon A_i), (2n)!\varepsilon A_i\right\} \tag{2.11}$$

where $A_i := \frac{\alpha_i + (2i+1)\beta_i}{\beta_i(2i+1)!}$.

- We shall prove that any $x \in \mathcal{X}(P_0)$, where $P_0 := (0,0,...0,x_0,x_0',0,...,0) \in E_{i0}$ and $x_0 = \varepsilon\eta$, does not egresses from the cone $\mathbf{K}$, for any $t_1 \leq 1$.

We shall first show that

$$0 \leq x(t) \leq \eta,\ \ 0 \leq t \leq 1.$$

Indeed in view of (2.4), let us assume that there exists $t^* \in (0,1]$ such that

$$0 \leq x(t) \leq \eta,\ 0 \leq t < t^*, \text{ and } \ x(t^*) = \eta.$$

Then by the Taylor's formula, assumption (A.1), (2.3) and (2.10), we get $\bar{t} \in (0,t^*)$ such that

$$\begin{aligned} \eta = x(t^*) \quad &= x_0\left[\tfrac{1}{(2i)!} + \tfrac{\alpha_i}{\beta_i}\tfrac{1}{(2i+1)!}\right] - \tfrac{1}{2n)!}f(\bar{t}, x(\bar{t})) \\ &\leq \varepsilon\eta\tfrac{(2i+1)\beta_i + \alpha_i}{(2i+1)!\beta_i} + \tfrac{1}{(2n)!}\mu x(\bar{t}) \\ &\leq \varepsilon\eta\tfrac{(2i+1)\beta_i + \alpha_i}{(2i+1)!\beta_i} + \tfrac{1}{(2n)!}\mu\eta. \end{aligned}$$

Consequently we obtain

$$\mu \geq (2n)!\left[1 - \varepsilon\frac{\alpha_i + (2i+1)\beta_i}{\beta_i(2i+1)!}\right]$$

contrary to the choice of μ at (2.1). Similarly if

$$0 \leq x(t) \leq \eta,\ 0 \leq t < t^*, \text{ and } \ x(t^*) = 0$$

then since $x(t) \leq \eta$, $0 \leq t < t^*$, as above

$$0 \geq \varepsilon\eta A_i - \frac{1}{(2n)!}\mu x(\bar{t}) \geq \varepsilon\eta A_i - \frac{1}{(2n)!}\mu\eta,$$

contrary to the choice of μ at (2.11).

Consider again the function $G_i(P) := \gamma_i x^{(2i)}(1) + \delta x^{(2i+1)}(1)$ defined above and then as above, by (2.10), we get

$$\begin{aligned} G_i(P) &= x_0 \left[\tfrac{\rho_i}{\beta_i} - \tfrac{\gamma_i}{(2n-2i)!}\tfrac{f(\hat{t},x(\hat{t}))}{x_0} - \tfrac{\delta_i}{(2n-2i-1)!}\tfrac{f(t^*,x(t^*))}{x_0}\right] \\ &\geq \varepsilon\eta\tfrac{\rho_i}{\beta_i} - \tfrac{\gamma_i}{(2n-2i)!}\mu x(\hat{t}) - \tfrac{\delta_i}{(2n-2i-1)!}\mu x(t^*) \\ &\geq \varepsilon\eta\tfrac{\rho_i}{\beta_i} - \tfrac{\gamma_i}{(2n-2i)!}\mu\eta - \tfrac{\delta_i}{(2n-2i-1)!}\mu\eta = \varepsilon\eta\tfrac{\rho_i}{\beta_i} - \mu\eta\tfrac{\gamma_i+(2n-2i)\delta_i}{(2n-2i)!}. \end{aligned}$$

Thus by (2.11), we conclude that there is a point $P_{i0} := (0, 0, ...0, x_0, y_0, 0, ..., 0) \in E_{i0}$ (with $x_0 = \varepsilon\eta < \frac{\beta_i(2i+1)!}{\alpha_i+(2i+1)\beta_i}\eta$) such that

$$G_i(P_{i0}) > 0.$$

Following similar steps (see also [19]), we may easily prove the next symmetric result.

Lemma 2.2. Assume (A.1), (A.3) and $(A.2_*)$ (i.e. the function f sublinear) hold. Then for each $i = 1, 2, .., n$

- there exists an $\eta_i > 0$ such that any trajectory $X(t, P_0)$ with $P_0 := (0, 0, ...0, x_0, x_0', 0, ..., 0) \in E_{i0}$ and $x_0 \leq \eta_i$, egresses from **K** through E_{i1}, at a time $t_1 \leq 1$.
- there exist an $H_i > 0$ and $P_1 := (0, 0, ...0, x_1, x_1', 0, ..., 0) \in E_{i0}$ with $x_1 \geq H_i$ such all trajectories $X(t, P)$ do not egress from **K** (they stay "asymptotic" in **K**) for all time $t \in [0, 1]$.

We shall study now the boundary value problem (1.5)-(1.6). Our main tool will be the Sperner's Lemma 1.4.

Theorem 2.3. Assume (A.1) and (A.3) hold. Then the boundary value problem (1.5)-(1.6) has a solution provided that the function f is superlinear (see $(A.2^*)$) or sublinear (see $(A.2_*)$). Furthermore there exist $0 < \eta < H$ such that

$$\eta \leq x^{(2i)}(t) \leq H, \quad 0 \leq t \leq 1, \quad (i = 0, 1, ..., n-1)$$

for any such solution.

Proof. Let us study the *superlinear* case. For $\rho = 0, 1, ..., n-1$, we set

$$R_k := \frac{(2k+1)\gamma_\rho\beta_k + \delta_\rho\alpha_k + \alpha_k\gamma_\rho}{(2k+1)\beta_k}, \; k = \rho, \rho+1, ..., n-1$$

and choose

$$K \geq \max_{0 \leq \rho \leq n-1} \left\{ \frac{(2n-2\rho)!}{\gamma_\rho + \delta_\rho(2n-2\rho)}, \frac{(2n-2\rho)!}{\gamma_\rho + \delta_\rho(2n-2\rho)} \sum_{k=\rho}^{n-1} \frac{R_k}{(2k)!} \right\}. \quad (2.12)$$

Then by assumption $(A.2^*)$ (superlinearity of f) there exists $H > 0$ such that

$$\min_{0 \leq t \leq 1} f(t, x) > Kx, \quad x \geq H.$$

Consider the vectors

$$e^{i+1} := \overrightarrow{P_{i0}, P_{i1}} \subseteq E_{i0}\ , \;\; i = 0, 1, ..., n-1,$$

where recall that $[P_{i0}, P_{i1}]$ is the segment in E_{i0}, defined in Lemma 2.1. Let S be the n-simplex spanned by the vertices $e^0 = 0$ and e^i, $1 \leq i \leq n$. As usual $[e^{i_0}e^{i_1}...e^{i_r}]$ denote the closed face of S spanned by the vertices $\{e^{i_0}, e^{i_1}, ..., e^{i_r}\}$. We choose here P_{i1} large enough, so that

$$\left\{ \begin{array}{l} \text{If an initial vector } v \text{ starts from } e^0 = 0 \text{ and ends on the face} \\ [e^{i_0}e^{i_1}...e^{i_r}] \text{ (which does not contain } e^0\text{), then any solution} \\ x(t, v) \text{ egresses from } \mathbf{K} \text{ for some } t_1 \leq 1. \end{array} \right. \quad (2.13)$$

Such a choice of P_{i1} is always possible. Indeed, let's assume that for every $P_{\rho 1}$ with $\rho \in \{i_0, i_1, ..., i_r\}$, there is $v \in [e^{i_0}e^{i_1}...e^{i_r}]$ and a solution $x(t, v)$ which remains asymptotic in $\mathbf{K}$, for $0 \leq t \leq 1$. Choose (and fix) a point $P_{\rho 1}$ large enough so that

$$v^* := \max\{v_\rho, v_{\rho+1}, ..., v_{n-1}\} = v_\rho > H_i \geq H.$$

Then since $v \in \mathbf{K} \cap [e^{i_0}e^{i_1}...e^{i_r}]$, we have

$$x^{(2j)}(0, v) = v_j \geq 0 \text{ and } x^{(2j+1)}(0, v) = v'_j \geq 0, \;\; j = 0, 1, ..., n-1. \quad (2.14)$$

Consequently, by the definition of

$$G_\rho(t,v) = \gamma_\rho x^{(2\rho)}(t,v) + \delta_\rho x^{(2\rho+1)}(t,v),\ 0 \le t \le 1,$$

we have

$$G_\rho(0,v) > 0.$$

Furthermore, by Taylor formula we get for some $\hat{t}, t^* \in [0,1]$

$$\begin{aligned} x^{(2\rho)}(1,v) \quad &= \textstyle\sum_{j=2\rho}^{2n-1} \frac{1}{j!} x^{(j)}(0) + \frac{1}{(2n-2\rho)!} x^{(2n)}(\hat{t}) \text{ and} \\ x^{(2\rho+1)}(1,v) \quad &= \textstyle\sum_{j=2\rho+1}^{2n-1} \frac{1}{j!} x^{(j)}(0) + \frac{1}{(2n-2\rho-1)!} x^{(2n)}(t^*). \end{aligned}$$

Also by the definition of the simplex $[e^{i_0}e^{i_1}...e^{i_r}]$, we still have

$$\alpha_k x^{(2k)}(0) = \beta_k x^{(2k+1)}(0),\ k = 0,1,...,n-1$$

and then an easy computation leads to

$$\begin{aligned} G_\rho(1;v) \quad &= \textstyle\sum_{k=\rho}^{n-1} \frac{x^{(2k)}(0)}{(2k)!} R_k - \frac{\gamma_\rho}{(2n-2\rho)!} f(\hat{t}, x(\hat{t})) - \frac{\delta_\rho}{(2n-2\rho-1)!} f(t^*, x(t^*)) \\ &= \textstyle\sum_{k=\rho}^{n-1} \frac{v_k}{(2k)!} R_k - \frac{\gamma_\rho}{(2n-2\rho)!} f(\hat{t}, x(\hat{t})) - \frac{\delta_\rho}{(2n-2\rho-1)!} f(t^*, x(t^*)) \end{aligned}$$

where we recall that

$$R_k := \frac{(2k+1)\gamma_\rho\beta_k + \delta_\rho\alpha_k + \alpha_k\gamma_\rho}{(2k+1)\beta_k},\ k = \rho, \rho+1, ..., n-1.$$

Thus

$$G_\rho(1;v) \le \quad v^* \textstyle\sum_{k=\rho}^{n-1} \frac{R_k}{(2k)!} - \frac{\gamma_\rho}{(2n-2\rho)!} f(\hat{t}, x(\hat{t})) - \frac{\delta_\rho}{(2n-2\rho-1)!} f(t^*, x(t^*)).$$

As in the proof of Lemma 2.1 (see (2.7)), if

$$v^* \le \min\{x(\hat{t}), x(t^*)\} \tag{2.15}$$

then by the choice K at (2.12), we easily conclude that

$$G_\rho(1;v) < 0,$$

contrary to the assumption that $x(t,v)$ remains asymptotic in **K** for all $t \in [0,1]$.

If in the contrary, assume that

$$\begin{cases} \text{there is a sequence } \{v_k\} \subseteq \mathbf{K} \cap [e^{i_0} e^{i_1} \dots e^{i_r}] : \\ \lim v_{k\rho} = \infty \text{ and } x(\hat{t}_k, v_k) \leq v^* \end{cases} \tag{2.16}$$

$(v_k = (v_{k0}, \dots, v_{k\rho}, \dots, v_{k(n-1)}))$ then, if $x_k(t) := x(t, v_k)$, $k = 1, 2, \dots$, by (2.16), the Taylor's formula and the monotonicity of $x_k(t)$, we get for some $\bar{t} \in (0, \hat{t})$,

$$\begin{aligned} x_k(\hat{t}_k) &= \textstyle\sum_{j=0}^{2n-1} \frac{1}{j!} x_k^{(j)}(0) + \frac{1}{(2n)!} x_k^{(2n)}(\bar{t}_k) \\ &\geq v_{k\rho} \tfrac{\hat{t}_k^{2\rho}}{(2\rho)!} [1 + \tfrac{\alpha_\rho}{(2\rho+1)\beta_\rho} \hat{t}_k] - \tfrac{1}{(2n)!} f(\bar{t}_k, x_k(\bar{t}_k)) \\ &\geq v_{k\rho} \tfrac{\hat{t}_k^{2\rho}}{(2\rho)!} [1 + \tfrac{\alpha_\rho}{(2\rho+1)\beta_\rho} \hat{t}_k] - N, \end{aligned}$$

where

$$N := \frac{1}{(2n)!} \max\{f(t, x) : 0 \leq t \leq 1, 0 \leq x \leq v^*\}.$$

As in (2.8), 0 cannot be an accumulation point of $\{\hat{t}_k\}$, since then we get

$$\lim x_k(\hat{t}_k) = \lim v_{k\rho} = +\infty,$$

contrary to (2.16). Therefore the choice of P_{i1} with property (2.13), is always possible.

Define now the sets $(\rho = 0, 1, \dots, n-1)$

$$\begin{cases} E_0 = cl\{v \in S : X(t, v) \text{ remains in } \mathbf{K}, \text{ for } 0 \leq t \leq 1 \text{ and} \\ E_\rho = cl\{v \in S : X(t, v) \text{ egresses from } \mathbf{K} \text{ through } E_{\rho 1} \text{ for } t_1 \leq 1\}. \end{cases}$$

In order to apply the Sperner's lemma, it is necessary to verify that

- the sets $\{E_\rho\}$ form a closed covering of S and
- $[e^{i_0} e^{i_1} \dots e^{i_r}] \subseteq \cup_{j=0}^{r} E_{i_j}$.

The closedness of E_ρ is obvious by the continuous dependence of solutions on their initial data. Further, their union covers S, because any trajectory either remains in $\mathbf{K}$ for all $0 \leq t \leq 1$ or else egresses from it, through some face E_{i1} for some $t_1 \leq 1$.

Let now $v \in [e^{i_0}e^{i_1}...e^{i_r}]$ (more precisely v starts from $e^0 = 0$ and ends on the face $[e^{i_0}e^{i_1}...e^{i_r}]$). Examine two cases:

If $0 \in \{i_0, i_1, ..., i_r\}$, then, if $x(t,v)$ remains in $\mathbf{K}$ for all $t \in [0,1]$, then $v \in E_0 \subseteq \cup_{j=0}^{r} E_{i_j}$. Otherwise, let us assume that $x(t,v)$ egresses from $\mathbf{K}$ through $E_{\rho 1}$, where $\rho \notin \{i_0, i_1, ..., i_r\}$. Then clearly

$$x^{(2\rho)}(0,v) = 0 = x^{(2\rho+1)}(0,v).$$

Therefore

$$x^{(2\rho)}(t,v)x^{(2\rho+1)}(t,v) > 0, \tag{2.17}$$

for all those $t \in (0,1]$ for which

$$x^{(2\rho+2)}(s,v) \neq 0,\ 0 < s \leq t. \tag{2.18}$$

Now by the definition of $G_\rho(t,v)$, it is clear (see Remark 1.3) that

$$G_{\rho+1}(t,v) \neq 0, \quad 0 < t \leq 1,$$

implies (2.18) and the last one verifies (2.17), i.e.

$$G_\rho(t,v) \neq 0, \quad 0 < t \leq 1.$$

Consequently the trajectory $x(t,v)$ egresses from the cone $\mathbf{K}$ through a plane E_{j1} with $j \neq \rho$, a contradiction. This results that $\rho \in \{i_0, i_1, ..., i_r\}$ and so $v \in E_\rho \subseteq \cup_{j=0}^{r} E_{i_j}$.

If $0 \notin \{i_0, i_1, ..., i_r\}$, then by choice of P_{i1} at (2.13), any solution $x(t,v)$ egresses from $\mathbf{K}$. Then as above it must egresses from $\mathbf{K}$ through a plane $E_{\rho 1}$, with $\rho \in \{i_0, i_1, ..., i_r\}$ and so we get once again $v \in E_\rho \subseteq \cup_{l=0}^{r} E_{i_l}$. Therefore $[e^{i_0}e^{i_1}...e^{i_r}] \subseteq \cup_{j=0}^{r} E_{i_j}$.

We can apply now the Sperner's Lemma 1.4, and thus we conclude that there exists a vector

$$v_0 \in \cap_{i=0}^{n} E_i$$

and so a solution $x = x(t;v)$ of differential equation (1.6), which by the definition of E_i, must satisfy the boundary conditions (1.5). Furthermore, by the definition of the cone $\mathbf{K}$ and since $v_0 \in E_0$, it is obvious that

$$(X(t), X'(t)) \in \mathbf{K}, \ 0 \leq t < 1$$

and so

$$\gamma X(t) + \delta X'(t) > 0, \ 0 \le t < 1.$$

This last result clearly means that

$$\gamma_i x^{(2i)}(t) + \delta_i x^{(2i+1)}(t) > 0, \ 0 \le t < 1, \ i = 0, 1, ..., n-1,$$

and since $\alpha_i x^{(2i)}(t) - \beta_i x^{(2i+1)}(t) > 0$, it follows by condition $\alpha_i \delta_i + \beta_i \gamma_i > 0$ (see also Remark 1.2) that

$$x^{(2i)}(t) > 0, \ 0 \le t \le 1, \ i = 0, 1, ..., n-1.$$

Furthermore, by Taylor formula, we get for some $\bar{t} \in [0,1]$

$$\begin{aligned} x^{(2\rho)}(t,v) &= \textstyle\sum_{j=2\rho}^{2n-1} \frac{1}{j!} x^{(j)}(0) + \frac{1}{(2n-2\rho)!} x^{(2n)}(\bar{t}) \text{ and} \\ &\le \textstyle\sum_{j=\rho}^{n-1} \frac{1}{j!} \left\{1 + \frac{\alpha_j}{(j+1)\beta_j}\right\} v_j \le \sum_{j=\rho}^{n-1} \frac{1}{j!} \left\{1 + \frac{\alpha_j}{(j+1)\beta_j}\right\} H_j \\ &\le \textstyle\sum_{j=0}^{n-1} \frac{1}{j!} \left\{1 + \frac{\alpha_j}{(j+1)\beta_j}\right\} H_j := H. \end{aligned}$$

because by above analysis, $v \notin E_0$ for arbitrary large v_j, say $v_j \ge H_j$. Similarly we may prove that

$$x^{(2\rho)}(t,v) \ge \eta, \ 0 \le t < 1$$

for some $\eta > 0$ and this completes the proof of theorem for the superlinear case.

A similar argument can be used to treat the *sublinear* case.

3 More Results

Consider now the boundary value problem

$$x^{(2n)}(t) = -f(t, x(t)), \quad 0 \le t \le 1. \tag{3.1}$$

$$\begin{cases} \alpha_i x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \\ \gamma_i x^{(2i)}(1) - \delta_i x^{(2i+1)}(1) = 0. \end{cases} \tag{3.2}$$

We are going to prove existence of a solution $x = x(t)$ of (3.1)-(3.2) such, that

$$x^{(2i)}(t) > 0 \quad \text{and } x^{(2i+1)}(t) > 0, \ 0 \le t \le 1, \ i = 0, 1, ..., n-1, \tag{3.3}$$

where still assume that the function $f \in C([0,1] \times [0,\infty), [0,\infty))$ is superlinear or sublinear and

$$(A.3^*) \qquad \rho_i^* := \beta_i\gamma_i + \alpha_i\gamma_i - \alpha_i\delta_i < 0, \;\; 0 \le i \le n-1.$$

Theorem 3.1. Assume (A.1), $(A.2_*)$ (or $(A.2^*)$) and $(A.3^*)$ hold. Then the boundary value problem (3.1)-(3.2) has a solution satisfying (3.3).

Proof. Let f be superlinear. As in the proof of Lemma 2.1, we can see that for any

$$K > \max\left\{(2n)!\left[\frac{\beta_i(2i+1)+\alpha_i}{\beta_i(2i+1)!}\right], \frac{(2n-2i)!\rho_i}{(\gamma_i + 2\delta_i(n-i))\beta_i}\right\}$$

there exists $H > 0$ such that

$$\min_{0\le t\le 1} f(t,x) > Kx, \quad x \ge H \tag{3.4}$$

and further there exists an $H_i > H$ such that for any $P_{i1} := (0,0,...0,x_1,x_1',0,...,0) \in E_{i0}$ with $x_1 \ge H_i$ every trajectory $x \in \mathcal{X}(P_{i1})$ satisfies

$$0 \le x(t) \le \tfrac{x_1}{(2i)!}\left[1 + \tfrac{\alpha_i}{\beta_i(2i+1)}\right], \;\; 0 \le t \le t_1 \;\; \text{and}$$
$$G_i(P_{i1}) = \gamma_i x^{(2i)}(t_1) + \delta_i x^{(2i+1)}(t_1) < 0.$$

for some $t_1 \in (0,1]$. Therefore

$$G_i^*(t_1, P_{i1}) := \gamma_i x^{(2i)}(t_1) - \delta_i x^{(2i+1)}(t_1) > 2\gamma_i x^{(2i)}(t_1) > 0$$

and this means that the solution $x \in \mathcal{X}(P_{i1})$ egresses now from the (new) cone

$$\mathbf{K}^* := \{(X,X') : X \ge 0 \text{ and } \gamma X - \delta X' \le 0\},$$

through the plane

$$E_{i1} := \{(X,X') : \gamma_i x_i - \delta_i x_i' = 0, \; x_i \ge 0, \; x_j = 0 = x_j', \; j \ne i\}$$

for a time $t_1 < 1$.

Similarly (see (2.11)) since $f_0 = 0$, for any

$$\mu < \min\left\{-\frac{\varepsilon\rho_i^*(2n-2i-1)!}{\beta_i\delta_i}, (2n)!(1-\varepsilon A_i), (2n)!\varepsilon A_i\right\}$$

there is $\eta > 0$ such that any $x \in \mathcal{X}(P_0)$, where $P_0 := (0, 0, ...0, x_0, x_0', 0, ..., 0) \in E_{i0}$ and $x_0 = \varepsilon\eta$, does not egresses from the cone $\mathbf{K}^*$ and further

$$0 \leq x(t) \leq \eta, \ \ 0 \leq t \leq 1.$$

Consequently by (2.10) and assumption (A.1) we get

$$\begin{aligned} G_i^*(P) \quad &= x_0 \left[\tfrac{\rho_i^*}{\beta_i} - \tfrac{\gamma_i}{(2n-2i)!} \tfrac{f(\hat{t}, x(\hat{t}))}{x_0} + \tfrac{\delta_i}{(2n-2i-1)!} \tfrac{f(t^*, x(t^*))}{x_0}\right] \\ &\leq \varepsilon\eta \tfrac{\rho_i^*}{\beta_i} + \tfrac{\delta_i}{(2n-2i-1)!} \mu x(t^*) \\ &\leq \varepsilon\eta \tfrac{\rho_i^*}{\beta_i} + \tfrac{\delta_i}{(2n-2i-1)!} \mu\eta < 0, \end{aligned}$$

and so the solution $x \in \mathcal{X}(P_0)$ remains asymptotic in $\mathbf{K}^*$ for all $t \leq 1$.

Now, Theorem 2.3 can readily applied (we must replace $\mathbf{K}$ by $\mathbf{K}^*$ and $G_\rho(1, .)$ by the corresponding $G_{\rho^*}(1, .)$, in order to verify (2.13).

Finally since the obtaining trajectory $(X(t), X'(t)) \in \mathbf{K}^*$, we get

$$\gamma X(t) - \delta X'(t) < 0, \ 0 \leq t \leq 1.$$

So we easily obtain inequalities (3.3) and this ends the proof.

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A Note on Stochastic Minimax Principle*

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(Submitted by: V. Sree Hari Rao)

Abstract

We study zero-sum stochastic differential games of fixed duration where the state is given by controlled (possibly) degenerate diffusions. Using the framework of relaxed strategies, we derive a stochastic minimax principle.

1 Introduction

We study a zero-sum stochastic differential game on the finite horizon where the state $X(\cdot)$ is an R^d-valued controlled degenerate diffusion given

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by

$$\begin{aligned} dX(t) &= b(t, X(t), u_1(t), u_2(t))\ dt + \sigma(X(t))\ dW(t), \quad t \in (0, T] \\ X(0) &= x, \end{aligned} \tag{1.1}$$

where $b : [0, T] \times \mathrm{R}^d \times U_1 \times U_2 \to \mathrm{R}^d$, U_1 and U_2 are action sets of players 1 and 2 respectively; $W(\cdot)$ is a standard d-dimensional Brownian motion; $\sigma : \mathrm{R}^d \to \mathrm{R}^{d\times d}$; $u_i : [0, T] \to U_i$ is progressively measurable with respect to the σ-field generated by $W(\cdot)$ which is the strategy of player $i, i = 1, 2$. Player 1 tries to maximize his expected payoff

$$R(x, u_1(\cdot), u_2(\cdot)) := E\Big[\int_0^T r(t, X(t), u_1(t), u_2(t))\ dt + g(X(T))\Big] \tag{1.2}$$

over his strategies $u_1(\cdot)$, whereas player 2 tries to minimize the same over his strategies $u_2(\cdot)$. Here r is the running payoff function and g is the terminal payoff function. Precise conditions on b, σ, g, u_1, u_2 will be given in next section. The stochastic differential game (SDG for short) has a value if

$$\inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(x, u_1(\cdot), u_2(\cdot)) = \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(x, u_1(\cdot), u_2(\cdot)).$$

A strategy $u_1^\star(\cdot)$ is said to be optimal for player 1 if

$$R(x, u_1^\star(\cdot), \tilde{u}_2(\cdot)) \geq \inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_2(\cdot)$ of player 2. Similarly, a strategy $u_2^\star(\cdot)$ is said to be optimal for player 2 if

$$R(x, \tilde{u}_1(\cdot), u_2^\star(\cdot)) \leq \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_1(\cdot)$ of player 1. A pair of optimal strategies for both players satisfies

$$R(x, u_1(\cdot), u_2^\star(\cdot)) \leq R(x, u_1^\star(\cdot), u_2^\star(\cdot)) \leq R(x, u_1^\star(\cdot), u_2(\cdot))$$

for any pair of strategies $(u_1(\cdot), u_2(\cdot))$ of the players. Thus $(u_1^\star(\cdot), u_2^\star(\cdot))$ constitutes a saddle point equilibrium. Conversely, a pair of saddle point strategies $(u_1^\star(\cdot), u_2^\star(\cdot))$ is clearly a pair of optimal strategies for both players.

In this note, we establish a (stochastic) minimax principle in the framework of relaxed strategies which gives necessary conditions for optimality. This framework is explained in the next section. Our paper is organized as follows. In Section 2, we introduce the basic notation and assumptions. The minimax principle is derived in Section 3.

2 Preliminaries

Let $U_i, i = 1, 2$, be given compact metric spaces and $\mathcal{M}_i = \mathcal{P}(U_i)$, the space of probability measures on U_i. Let $T > 0$ be fixed. Let

$$\bar{b} : [0,T] \times \mathrm{R}^d \times U_1 \times U_2 \to \mathrm{R}^d$$

and

$$\sigma : [0,T] \times \mathrm{R}^d \to \mathrm{R}^{d\times d}.$$

We assume that
(A1) the functions $\bar{b}$ and σ are continuous and there exists a constant $C_1 > 0$ such that

$$|\bar{b}(t,x,u_1,u_2) - \bar{b}(s,y,u_1,u_2)| + |\sigma(t,x) - \sigma(s,y)| \leq C_1(|t-s| + |x-y|)$$

for all $(u_1, u_2) \in U_1 \times U_2$ and $(t,x), (s,y) \in [0,T] \times \mathrm{R}^d$. Define

$$b : [0,T] \times \mathrm{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathrm{R}^d$$

by

$$b(t,x,\mu_1,\mu_2) = \int_{U_2}\int_{U_1} \bar{b}(t,x,u_1,u_2)\ \mu_1(du_1)\ \mu_2(du_2).$$

The state of the system $X(\cdot)$ evolves according to the controlled stochastic differential equation of Ito type

$$\begin{aligned} dX(t) &= b(t, X(t), \mu_1(t), \mu_2(t))\ dt + \sigma(t, X(t))\ dW(t), \quad t \in (0,T], \\ X(0) &= x \in \mathrm{R}^d. \end{aligned} \tag{2.1}$$

Here $W(\cdot)$ is a standard d-dimensional Wiener process; $\mu_i(\cdot)$ is an $\mathcal{M}_i$-valued process which is progressively measurable with respect to σ-fields

$\mathcal{F}_t^W := \sigma(W(t') : 0 \leq t' \leq t)$. The process $\mu_i(\cdot)$ is called an (admissible) relaxed strategy for player i. Let $\mathcal{A}_i$ denote the set of all admissible strategies for player i. A relaxed strategy $\mu_i(\cdot)$ of player i is called a pure strategy if $\mu_i(\cdot)$ is a Dirac measure, i.e., $\mu_i(\cdot) = \delta_{u_i(\cdot)}$, where $u_i(\cdot)$ is a U_i-valued process.

Let $\bar{r} : [0,T] \times \mathrm{R}^d \times U_1 \times U_2 \to \mathrm{R}$ be the running payoff function. If x is the state at time t and the players choose actions $(u_1, u_2) \in U_1 \times U_2$, then player 1 receives a payoff $\bar{r}(t, x, u_1, u_2)$ from player 2 at time t. Let $g : \mathrm{R}^d \to \mathrm{R}$ denote the terminal payoff function.

We assume that:

(A2) (i) the functions $\bar{r}$ and g are bounded and continuous;

(ii) there exist constants $C_2 > 0$, $C_3 > 0$, such that

$$\begin{aligned} |\bar{r}(t,x,u_1,u_2) - \bar{r}(s,y,u_1,u_2)| &\leq C_2(|t-s| + |x-y|), \\ |g(x) - g(y)| &\leq C_3|x-y|, \end{aligned}$$

for all $t, s \in [0,T]$, $x, y \in \mathrm{R}^d$, $(u_1, u_2) \in U_1 \times U_2$.

Let $r : [0,T] \times \mathrm{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathrm{R}$ be defined by

$$r(t,x,\mu_1,\mu_2) = \int_{U_2}\int_{U_1} \bar{r}(t,x,u_1,u_2)\ \mu_1(du_1)\ \mu_2(du_2).$$

When the initial state of the system at time t is x and the players use relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$, then the payoff of player 1 is given by

$$R(x,\mu_1(\cdot),\mu_2(\cdot)) = E\left[\int_0^T r(s, X(s), \mu_1(s), \mu_2(s))\ ds + g(X(T))\right], \tag{2.2}$$

where $X(\cdot)$ is given by (2.1). A relaxed strategy $\mu_1^\star(\cdot) \in \mathcal{A}_1$ is said to be optimal for player 1, if

$$R(x,\mu_1^\star(\cdot),\mu_2(\cdot)) \geq \inf_{\mu_2(\cdot)\in\mathcal{A}_2}\ \sup_{\mu_1(\cdot)\in\mathcal{A}_1}\ R(x,\mu_1(\cdot),\mu_2(\cdot)),$$

for any $\mu_2(\cdot) \in \mathcal{A}_2$. Similarly, a relaxed strategy $\mu_2^\star(\cdot) \in \mathcal{A}_2$ is said to be optimal for player 2, if

$$R(t,x,\mu_1(\cdot),\mu_2^\star(\cdot)) \leq \sup_{\mu_1(\cdot)\in\mathcal{A}_1}\ \inf_{\mu_2(\cdot)\in\mathcal{A}_2}\ R(x,\mu_1(\cdot),\mu_2(\cdot)),$$

for any $\mu_1(\cdot) \in \mathcal{A}_1$. Thus a pair of optimal strategies constitute a saddle point equilibrium.

Remark 2.1. Since $\mu_i(\cdot)$ is progressively measurable with respect to the σ-field generated by $W(\cdot)$, there exists a progressively measurable function f_i such that $\mu_i(t) = f_i(t, W(\cdot)), 0 \leq t \leq T$. Thus the player 1 chooses the function f_1, whereas the player 2 chooses f_2. This way the noncooperative nature of the game is preserved.

3 Stochastic Minimax Principle

In this section, we derive a stochastic minimax principle. We make the following additional assumptions.

(A3) (i) For each $(t, u_1, u_2) \in [0, T] \times U_1 \times U_2$, the functions $\bar{b}(t, \cdot, u_1, u_2)$, $\bar{r}(t, \cdot, u_1, u_2)$ and $\sigma(\cdot)$ are continuously differentiable.

(ii) The function g is continuously differentiable.

Let $(\mu_1^\star(\cdot), \mu_2^\star(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ be a pair of optimal strategies and $X^\star(\cdot)$ the process corresponding to $(\mu_1^\star(\cdot), \mu_2^\star(\cdot))$ with initial condition $X^\star(0) = x$. We define the following "adjoint" process $p(\cdot)$ given by

$$\begin{aligned} dp(t) &= -b_x'(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))\, p(t)\, dt \\ &\quad - D_x r(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))\, dt \\ p(T) &= D_x g(X^\star(T)), \end{aligned} \tag{3.1}$$

where b_x denotes the Jacobian matrix and b_x', its adjoint. Let $\varepsilon > 0$ and $\mu_1(\cdot) \in \mathcal{A}_1$. Let $X^\varepsilon(\cdot)$ denote the process

$$dX^\varepsilon(t) = b(t, X^\varepsilon(t), \mu_1^\varepsilon(t), \mu_2^\star(t))\, dt + \sigma(t, X^\varepsilon(t))\, dW(t), \quad X^\varepsilon(0) = x, \tag{3.2}$$

where $\mu_1^\varepsilon(\cdot) = \mu_1^\star(\cdot) + \varepsilon(\mu_1(\cdot) - \mu_1^\star(\cdot))$. Then we can prove the following lemma by invoking Gronwall's lemma. We omit the details.

Lemma 3.1. Assume (A1) - (A3). Let $Z(\cdot)$ denote the process given by

$$\begin{aligned} dZ(t) &= b_x(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))\, Z(t)\, dt + \sigma_x(X^\star(t))\, Z(t)\, dW(t) \\ &\quad + (b(t, X^\star(t), \mu_1(t), \mu_2^\star(t)) - b(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))\, dt, \\ Z(0) &= 0 \end{aligned} \tag{3.3}$$

Then

$$E\Big|\frac{1}{\varepsilon}\int_0^T (r(s, X^\varepsilon(s), \mu_1^\varepsilon(s), \mu_2^\star(s)) - r(s, X^\star(s), \mu_1^\star(s), \mu_2^\star(s)))\ ds$$
$$-\int_0^T [D_x r(s, X^\star(s), \mu_1^\star(s), \mu_2^\star(s)) \cdot Z(s) + r(s, X^\star(s), \mu_1(s), \mu_2^\star(s))$$
$$-r(s, X^\star(s), \mu_1^\star(s), \mu_2^\star(s))]ds\Big| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Let $G : [0, T] \times \mathrm{R}^d \times \mathrm{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathrm{R}$ be defined by

$$G(t, x, p, \mu_1, \mu_2) = b(t, x, \mu_1, \mu_2) \cdot p + r(t, x, \mu_1, \mu_2).$$

We, now, prove the following minimax principle.

Theorem 3.2. Assume (A1) - (A3). Let $(\mu_1^\star(\cdot), \mu_2^\star(\cdot), X^\star(\cdot))$ be as above. Then for a.e. $t \in [0, T]$,

$$\begin{aligned}
\min_{\mu_2 \in \mathcal{M}_2} \max_{\mu_1 \in \mathcal{M}_1} \quad & E[G(t, X^\star(t), p(t), \mu_1, \mu_2)] \\
& = \max_{\mu_1 \in \mathcal{M}_1} E[G(t, X^\star(t), p(t), \mu_1, \mu_2^\star(t))] \\
& = \max_{\mu_1 \in \mathcal{M}_1} \min_{\mu_2 \in \mathcal{M}_2} E[G(t, X^\star(t), p(t), \mu_1, \mu_2)] \\
& = \min_{\mu_2 \in \mathcal{M}_2} E[G(t, X^\star(t), p(t), \mu_1^\star(t), \mu_2)].
\end{aligned} \tag{3.4}$$

Proof. Using Lemma 3.1, we have

$$\frac{d}{d\varepsilon} R(x, \mu_1^\varepsilon(\cdot), \mu_2^\star(\cdot))\Big|_{\varepsilon=0}$$
$$= E\Big[\int_0^T \Big[D_x r(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t)) \cdot Z(t) + r(t, X^\star(t), \mu_1(t), \mu_2^\star(t)) -$$
$$-r(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))\Big] dt + D_x g(X^\star(T)) \cdot Z(T)\Big]. \tag{3.5}$$

Now

$$d(p(t) \cdot Z(t)) = Z(t) \cdot dp(t) + p(t).dZ(t).$$

Integrating the above from 0 to T, taking expectation and then rearranging the terms using (3.1) and (3.3), it follows that

$$E\Big[\int_0^T D_x r(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t)) \cdot Z(t)\ dt + D_x g(X^\star(T)) \cdot Z(T)\Big]$$
$$= E\Big[\int_0^T (b(t, X^\star(t), \mu_1(t), \mu_2^\star(t)) - b(t, X^\star(t), \mu_1^\star(t), \mu_2^\star(t))) \cdot p(t)\ dt\Big].$$

Hence by (3.5), we obtain

$$\begin{aligned}\frac{d}{d\varepsilon}R(x,\mu_1^{\varepsilon}(\cdot),\mu_2^{\star}(\cdot))\Big|_{\varepsilon=0} \;=\;& E\Big[\int_0^T \Big\{G(t,X^{\star}(t),p(t),\mu_1(t),\mu_2^{\star}(t)) - \\ & -G(t,X^{\star}(t),p(t),\mu_1^{\star}(t),\mu_2^{\star}(t))\Big\}\,dt\Big]. \quad (3.6)\end{aligned}$$

Since $(\mu_1^{\star}(\cdot),\mu_2^{\star}(\cdot))$ is an optimal pair, we have

$$\begin{aligned}E\Big[\int_0^T &\Big\{G((t,X^{\star}(t),p(t),\mu_1(t),\mu_2^{\star}(t)) \\ &-G(t,X^{\star}(t),p(t),\mu_1^{\star}(t),\mu_2^{\star}(t))\Big\}\,dt\Big] \le 0. \quad (3.7)\end{aligned}$$

Let $\mu_1(\cdot) \in \mathcal{A}_1$ be given by

$$\mu_1(s) = \begin{cases} \mu_1, & s \in (t, t+\varepsilon) \\ \mu_1^{\star}(s), & s \in [0,t] \cup [t+\varepsilon, T] \end{cases}$$

where $\mu_1 \in \mathcal{M}_1$ is arbitrary. Since $\varepsilon > 0$ is arbitrary, by (3.7), it follows that for any $\mu_1 \in \mathcal{M}_1$, a.e. t

$$E[G(t,X^{\star}(t),p(t),\mu_1,\mu_2^{\star}(t))] \;\le\; E[G(t,X^{\star}(t),p(t),\mu_1^{\star}(t),\mu_2^{\star}(t))]. \quad (3.8)$$

Similarly, we can show that for any $\mu_2 \in \mathcal{M}_2$, a.e. t

$$E[G(t,X^{\star}(t),p(t),\mu_1^{\star}(t),\mu_2^{\star}(t))] \;\le\; E[G(t,X^{\star}(t),p(t),\mu_1^{\star}(t),\mu_2)]. \quad (3.9)$$

From (3.8) and (3.9), the desired result follows.

Remark 3.3. We refer to [1], [2] for stochastic maximum principle for stochastic optimal control problem. Here we have adapted the arguments in [1]. There is, however, a crucial technical difference. Note that in [1], the maximum principle involves the derivative with respect to the control variable, whereas in the framework of relaxed strategies the derivative term with respect to the control variable does not appear. See [3] for a detailed discussion on stochastic maximum principle.

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Delay Independent Stability for Additive Neural Networks

Sue Ann Campbell

Abstract

An additive neural network model consisting of n elements with arbitrary connections, nonlinear activation functions and multiple time delays is considered. Necessary and sufficient conditions for the delay independent local asymptotic stability of equilibrium solutions are determined. Slightly stronger conditions are shown to ensure the delay independent global stability of an equilibrium solution. These results and other delay independent stability results in the literature are compared.

1. Introduction

In 1978, Grossberg [21] considered the equation

$$\dot{u}_i = d_i(u_i)\left[-b_i(u_i) + \sum_{j=1}^{n} a_{ij} g_j(u_j)\right], \tag{1.1}$$

which, for appropriate choices of the functions d_i, b_i, g_i encompasses a large variety of biological models, including several types of neural networks. In [21] and subsequent articles [9, 22] Grossberg et al. studied pattern formation

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and stability in this model. In 1984, Hopfield [26] considered a special case of model (1.1)

$$\dot{u}_i = -b_i u_i + \sum_{j=1}^{n} a_{ij} g_j(u_j), \tag{1.2}$$

as a continuous extension of a discrete, two state neural network model he had previously studied [25]. In equation (1.2), which we shall call the additive neural network model, u_i can be thought of as the mean soma potential of the ith neuron. Then each neuron is represented by a linear electrical circuit consisting of a resistor and capacitor, and is connected to the other neurons via nonlinear (usually sigmoidal) activation functions. The studies outlined above assume instantaneous updating of each neuron and instantaneous communication between the neurons, and thus the models are systems of ordinary differential equations.

Early work of Grossberg [16, 17, 18] considered the effect of including discrete time delays in neural network models to represent the signal propagation time between neurons. These models consist of equations for the neural activity, such as (1.2), with time delays in the connection terms, supplemented by equations modelling the variation of the synaptic weights, a_{ij}, due to learning. Although quite complicated systems of delay differential equations, results on the learning and pattern recognition properties of these networks were obtained. These are summarized and extended in [19, 20].

In 1989, Marcus and Westervelt [31] considered the simpler model of just (1.2) with time delays in the connection terms. The analysis of this model is still quite complicated, and thus [30, 31] and much of the initial work on these delayed, additive neural networks considered the situation where all the time delays in the network are the same. Examples include the work of Bélair et al. [4, 5, 6], Gopalsamy and Leung [15], Ye et al. [42] and Pakdaman et al. [33] (see also references therein).

Preliminary work on (1.2) with multiple time delays simplified the analysis by either restricting the size of the network (e.g., [32, 34, 37]), or considering networks with simple architectures (e.g., [2, 8, 35]). Recent work has focussed on the global stability of equilibrium points of an arbitrary sized network with different time delays in each connection term. These articles generally formulate conditions on the connection matrix under which there is global stability either independent of all the delays, as in Gopalsamy and He [13, 14], or under additional conditions on some delays, as in van den Driessche et al. [40, 41] and Ye et. al. [43]. (This last article considered the general model

(1.1) with time delays in the connection terms). By contrast, the articles of Hofbauer and So [24] and of So, Tang and Zou [38] formulate conditions related to the linear asymptotic stability of additive neural networks with multiple time delays.

The purpose of this article is to extend the results of [14] and [24] and to compare some of the delay independent and delay dependent criteria for stability found in the literature. We thus consider the nonlinear delay differential equation

$$\dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t - \tau_{ij})) + J_i, \tag{1.3}$$

where the RC decay constants b_i are positive and the time delays τ_{ij} are nonnegative. We will allow the connection coefficients a_{ij} to be arbitrary real numbers, with positive (negative) numbers representing excitatory (inhibitory) connections. Similarly, the neuron inputs, J_i, can be arbitrary real numbers. We will assume that the nonlinear functions g_j satisfy

(H1) $g_j \in C^2(R), \qquad g_j'(u) > 0, \qquad \sup_{u \in R} g_j'(u) = g_j'(0) = 1,$

(H2) $g_j(0) = 0, \qquad \lim_{u \to \pm\infty} g_j(u) = \pm 1.$

The function $g(u) = \tanh(u)$, which is commonly used in the model 1.3, satisfies these conditions.

We will consider initial value problems consisting of (1.3) subject to the conditions

$$u_i(\theta) = \phi_i(\theta), \; h \le \theta \le 0, \tag{1.4}$$

where $h = \max_{i,j}\{\tau_{ij}\}$ and the ϕ_i are continuous. The local existence and uniqueness of solutions is then guaranteed (see e.g., the book of Hale and Lunel [23]).

The plan for the article is as follows. In Section 2 we consider the linear stability analysis of equilibrium points of (1.3), giving necessary and sufficient conditions for them to be asymptotically stable for all $\tau_{ij} \geq 0$. In Section 3 we show how related conditions give delay independent global stability of an equilibrium point. In Section 4 we compare our results and those of [14, 38] and [40] and in Section 5 we summarize our results and draw some conclusions.

2. Linear Stability Analysis

Equilibrium points of (1.3) are solutions $u(t) = u^*, t \geq -h$. We will study the existence of such equilibrium points in Section 3. Assuming that such an equilibrium point exists one can translate it to the origin via the transformation $u(t) = u^* + x(t)$, yielding the equations

$$\dot{x}_i = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} \left[g_j(x_j(t - \tau_{ij}) + u_j^*) - g_j(u_j^*) \right]. \tag{2.1}$$

Due to (H1) we can use the linearization

$$\dot{x}_i = -b_i x_i(t) + \sum_{j=1}^{n} c_{ij} x_j(t - \tau_{ij}), \tag{2.2}$$

where $c_{ij} = a_{ij} g_j'(u_j^*)$, to study the asymptotic stability of the trivial solution of (2.1) and hence of the equilibrium point u^* of (1.3). To do this we consider the roots of the characteristic equation,

$$\Delta(\lambda) \stackrel{\text{def}}{=}$$

$$\det \begin{pmatrix} -b_1 + c_{11}e^{-\lambda\tau_{11}} - \lambda & c_{12}e^{-\lambda\tau_{12}} & \cdots & c_{1n}e^{-\lambda\tau_{1n}} \\ c_{21}e^{-\lambda\tau_{21}} & -b_2 + c_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & c_{2n}e^{-\lambda\tau_{2n}} \\ & \vdots & & \\ c_{n1}e^{-\lambda\tau_{n1}} & c_{n2}e^{-\lambda\tau_{n2}} & \cdots & -b_n + c_{nn}e^{-\lambda\tau_{nn}} - \lambda \end{pmatrix} = 0, \tag{2.3}$$

which correspond to nontrivial solutions of (2.2) of the form $x(t) = e^{\lambda t}v$.

Consider the matrices $B = \mathrm{diag}(b_1, b_2, \ldots, b_n)$, $C = (c_{ij})$ and $|C| = (|c_{ij}|)$ associated with (2.2) and define the matrices $K = -B + C$ and $\widehat{K} = -B + |C|$. In the following we will determine conditions on the matrices K and $\widehat{K}$ which are necessary and sufficient for all the roots of (2.3) to have negative real parts, independent of the size of the delays τ_{ij}. It follows that these will be necessary and sufficient conditions for the trivial solution of eq. (2.2) to be asymptotically stable and sufficient conditions for the equilibrium point $u = u^*$ of nonlinear system (1.3) to be locally asymptotically stable (see the books of Kolmanovskii and Nosov [28], Stépán [39], or Hale and Lunel [23]). Our conditions are a slight generalization of those presented in [24], and our proofs similar to theirs. The main difference is that we allow delays in the self connection terms, i.e., the τ_{ii} are not necessarily all zero.

Lemma 2.1. $\lambda = 0$ is a solution of equation (2.3) if and only if $\det K = 0$.

Proof. The proof follows immediately from the fact that $\Delta(0) = \det K$.

Before proceeding further, we state some standard results from Matrix Theory.

Definition 2.2. Let A be an $n \times n$ matrix whose off-diagonal elements are all nonpositive. We shall call A an M-matrix if all its principal minors are nonnegative.

There are many equivalent properties which can be used to define an M-matrix (see Fiedler [11, Theorem 5.3]), we have chosen the one most useful in our analysis. Note that our definition allows an M-matrix to be singular. In some texts [11] such matrices are referred to as class K_0 matrices. In [24] such matrices are called *weakly diagonally dominant*.

Lemma 2.3. If $-\widehat{K}$ is an M-matrix and $\det K \neq 0$ then the all the roots of (2.3) have negative real parts for all $\tau_{ij} \geq 0, 1 \leq i, j \leq n$.

Proof. First consider $-\widehat{K}$ irreducible. Then by [11, Theorem 5.9], there exist $\gamma_i > 0,\ i = 1, \ldots, n$ such that

$$\widehat{k}_{ii}\gamma_i + \sum_{j \neq i} \hat{k}_{ij}\gamma_j \leq 0, \qquad \text{for } i = 1, \ldots, n.$$

That is

$$(-b_i + |c_{ii}|)\gamma_i + \sum_{j \neq i} |c_{ij}|\gamma_j \leq 0, \qquad \text{for } i = 1, \ldots, n. \tag{2.4}$$

Let λ be a root of (2.3). Then λ is an eigenvalue of the matrix $D = (d_{ij})$, where $d_{ii} = -b_i + c_{ii}e^{-\lambda\tau_{ii}}$ and $d_{ij} = c_{ij}e^{-\lambda\tau_{ij}}$. Applying Geršgorin's theorem (cf. Lancaster and Tismenetsky [29, p. 371]) to the matrix $\widehat{D} = (\gamma_i^{-1} d_{ij} \gamma_j)$ we know that each eigenvalue, $\hat{\lambda}$, of $\widehat{D}$ satisfies

$$|\hat{\lambda} - d_{ii}| \leq \sum_{j \neq i} \gamma_i^{-1} |d_{ij}| \gamma_j,$$

for at least one $i \in \{1, 2, \ldots, n\}$. Now D is similar to $\widehat{D}$ so they have the same eigenvalues. Thus for each eigenvalue λ of D there is an i such that

$$\text{Re}(\lambda) \leq \text{Re}(d_{ii}) + \sum_{j \neq i} \gamma_i^{-1} |d_{ij}| \gamma_j.$$

Now suppose that $\Re(\lambda) \geq 0$. Then $\mathrm{Re}(d_{ii}) \leq -b_i + |c_{ii}|$, $|d_{ij}| \leq |c_{ij}|$ and

$$\mathrm{Re}(\lambda) \leq -b_i + |c_{ii}| + \sum_{j \neq i} \gamma_i^{-1} |c_{ij}| \gamma_j \tag{2.5}$$

Hence

$$\begin{aligned} \gamma_i \mathrm{Re}(\lambda) &\leq (-b_i + |c_{ii}|)\gamma_i + \sum_{j \neq i} |c_{ij}| \gamma_j \\ &\leq 0, \end{aligned}$$

which implies $\mathrm{Re}(\lambda) \leq 0$. Now equality occurs in (2.5) only when λ is real and $\det K \neq 0$ precludes $\lambda = 0$, hence we have a contradiction. We conclude that $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of D.

Now consider $-\widehat{K}$ reducible. Then there exists a permutation matrix P such that

$$P(-\widehat{K})P^T = \begin{bmatrix} \widetilde{K}_{11} & 0 & 0 & \cdots & 0 \\ \widetilde{K}_{21} & \widetilde{K}_{22} & 0 & \cdots & 0 \\ & & \vdots & & \\ \widetilde{K}_{n1} & \widetilde{K}_{n2} & \widetilde{K}_{n3} & \cdots & \widetilde{K}_{nn} \end{bmatrix},$$

where each $\widetilde{K}_{ii}$ is square and either irreducible or a 1×1 null matrix (cf. Berman and Plemmons [7, p. 39]). Since the permutations won't change the principle minors, each $\widetilde{K}_{ii}$ will be an M-matrix.

Let λ be a root of (2.3) and define D as above. Then PDP^T will also be lower block diagonal, with blocks $\widetilde{D}_{ii}$ corresponding to the $\widetilde{K}_{ii}$. Thus λ is root of

$$\begin{aligned} \Delta(\lambda) &= \det(D - \lambda I) \\ &= \pm \det(PDP^T - \lambda I) \\ &= \pm \prod_i \det(\widetilde{D}_{ii} - \lambda I). \end{aligned}$$

The rest of the proof follows by applying the argument for the irreducible case to each block, $\widetilde{K}_{ii}$.

Lemma 2.4. If $\det(-\widehat{K}) < 0$, then there exists delays $\tau_{ij} \geq 0$ such that (2.3) has a root λ with $\mathrm{Re}(\lambda) > 0$.

Proof. Consider the function

$$F_\epsilon(z) = \det \begin{pmatrix} -b_1 + c_{11}e^{-z\eta_{11}} - \epsilon z & c_{12}e^{-z\eta_{12}} & \cdots & c_{1n}e^{-z\eta_{1n}} \\ c_{21}e^{-z\eta_{21}} & -b_2 + c_{22}e^{-z\eta_{22}} - \epsilon z & \cdots & c_{2n}e^{-z\eta_{2n}} \\ \vdots & \vdots & & \vdots \\ c_{n1}e^{-z\eta_{n1}} & c_{n2}e^{-z\eta_{n2}} & \cdots & -b_n + c_{nn}e^{-z\eta_{nn}} - \epsilon z \end{pmatrix}$$

where

$$\eta_{ij} = \begin{cases} \frac{1}{2}, & \text{if } c_{ij} < 0 \\ 1, & \text{if } c_{ij} \geq 0. \end{cases}$$

For $z = x + 2\pi i$, where x is real, $F_0(z)$ becomes

$$D(x) = \det \begin{pmatrix} -b_1 + |c_{11}|e^{-x\eta_{11}} & |c_{12}|e^{-x\eta_{12}} & \cdots & |c_{1n}|e^{-x\eta_{1n}} \\ |c_{21}|e^{-x\eta_{21}} & -b_2 + |c_{22}|e^{-x\eta_{22}} & \cdots & |c_{2n}|e^{-x\eta_{2n}} \\ \vdots & \vdots & & \vdots \\ |c_{n1}|e^{-x\eta_{n1}} & |c_{n2}|e^{-x\eta_{n2}} & \cdots & -b_n + |c_{nn}|e^{-x\eta_{nn}} \end{pmatrix}.$$

By assumption, $(-1)^n D(0) = \det(-\widehat{K}) < 0$. Further $(-1)^n \lim_{x\to\infty} D(x) = b_1 b_2 \cdots b_n > 0$. Hence, by the Intermediate Value Theorem, there exists $\hat{x} > 0$ such that $D(\hat{x}) = 0$ and $\hat{z} = \hat{x} + 2\pi i$ is a zero of F_0. It follows from Rouché's Theorem (see Ahlfors [1, p. 153]) that the analytic function F_ϵ has a zero $\hat{z}(\epsilon)$ near $\hat{z}$ for small $\epsilon > 0$. Clearly, $\lambda = \hat{z}(\epsilon)\epsilon$ and $\tau_{ij} = \dfrac{\eta_{ij}}{\epsilon}$ satisfy (2.3), with $\text{Re}(\lambda) > 0$.

Lemma 2.5. If $\det K \neq 0$ and $-\widehat{K}$ is not an M-matrix, then there exists delays $\tau_{ij} \geq 0$ such that (2.3) has a root λ with $\text{Re}(\lambda) > 0$.

Proof. Since $-\widehat{K}$ is not an M-matrix, it follows that some principle minor of order m of $-\widehat{K}$ is negative. For notational convenience, we will assume that $\det(-\widehat{K}_m) < 0$ where $\widehat{K}_m$ is the $m \times m$ submatrix obtained by keeping the first m rows and columns of $\widehat{K}$; the argument is the same for any principle minor. Applying Lemma 2.4 to $\widehat{K}_m$ shows that the characteristic equation corresponding to this submatrix, i.e.,

$$\Delta_m(\lambda) \stackrel{\text{def}}{=} \det \begin{pmatrix} -b_1 + c_{11}e^{-\lambda\tau_{11}} - \lambda & c_{12}e^{-\lambda\tau_{12}} & \cdots & c_{1m}e^{-\lambda\tau_{1m}} \\ c_{21}e^{-\lambda\tau_{21}} & -b_2 + c_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & c_{2m}e^{-\lambda\tau_{2m}} \\ & \vdots & & \\ c_{m1}e^{-\lambda\tau_{m1}} & c_{m2}e^{-\lambda\tau_{m2}} & \cdots & -b_m + c_{mm}e^{-\lambda\tau_{mm}} - \lambda \end{pmatrix} = 0,$$

has a root $\hat{\lambda}$ with $\text{Re}(\hat{\lambda}) > 0$.

Applying Laplace's Theorem [29, p. 37], $\Delta(\lambda)$ can be written

$$\Delta(\lambda) = \Delta_m(\lambda)\Big[P(\lambda, e^{-\lambda\tau_{ii}}) + Q(\lambda, e^{-\lambda\tau_{ij}})\Big] + R(\lambda, e^{-\lambda\tau_{ij}}),$$

where P, Q, R are quasipolynomials with

$$P(\lambda) = \prod_{i=m}^{n}(-b_i + c_{ii}e^{-\lambda\tau_{ii}} - \lambda)$$

and $|Q|$ and $|R|$ can be made arbitrarily small, for $\lambda > 0$, by taking τ_{ij}, $(m+1) \leq i, j \leq n$ large enough. Thus by Rouché's Theorem (cf. [10, p. 247]) (2.3) also has a root with positive real part.

Theorem 2.6. The trivial solution of equation (2.2) is asymptotically stable for all delays $\tau_{ij} \geq 0$ if and only if $-\widehat{K}$ is an M-matrix and $\det K \neq 0$.

Proof. The trivial solution of (2.2) will be asymptotically stable if and only if all roots λ of (2.3) have negative real parts. Thus necessity follows from Lemmas 2.1 and 2.5 and sufficiency follows from Lemma 2.3.

If $a_{ij} > 0$ for all i, j then the conditions of Theorem 2.6 reduce to: $-K = -\widehat{K}$ is a nonsingular M-matrix. (See Definition 3.1 below.)

Corollary 2.7. Let $\det K \neq 0$. Then the equilibrium solution u^* of (1.3) is locally asymptotically stable for all delays $\tau_{ij} \geq 0$ if and only if $-\widehat{K}$ is an M-matrix.

Proof. The sufficiency follows directly from Theorem 2.6 and standard results on nonlinear delay differential equations [23]. The necessity follows from Lemma 2.5.

The result for (1.3) is slightly weaker than that for (2.2) as the situation when $\det K = 0$ will depend on the exact nonlinearities in the model.

3. Global Stability

In this section we will show, in a manner similar to that in [14], that conditions slightly stronger than those of Theorem 2.6 imply global stability of an equilibrium point of the nonlinear equation (1.3). To begin, we introduce the following definition.

Definition 3.1. Let A be an $n \times n$ matrix whose off-diagonal elements are all nonpositive. We shall call A a nonsingular M-matrix if all its principal minors are positive.

Consider the matrices $B = \text{diag}(b_1, b_2, \ldots, b_n)$ and $A = (a_{ij})$ associated with (1.3). Define the matrices $\mathcal{K}$ and $\widehat{\mathcal{K}}$ via $\mathcal{K} = -B + A$ and $\mathcal{K} = -B + |A|$, where $|A|$ is the matrix with elements $|a_{ij}|$.

Proposition 3.2. Assume (H1) and (H2) hold. If $-\widehat{\mathcal{K}}$ is a nonsingular M-matrix then (1.3) has a unique equilibrium point.

Proof. If $-\widehat{\mathcal{K}}$ is a nonsingular M-matrix then so is its transpose. Thus by a well-known Theorem ([7, Theorem 62.3], ([11, Theorem 5.1]) there are constants $\gamma_j > 0,\ j = 1, \ldots, n$ such that

$$-b_j\gamma_j + \sum_{i=1}^{n} |a_{ij}|\gamma_i < 0, \quad j = 1, \ldots, n. \tag{3.1}$$

Hence it follows that

$$\beta \overset{\text{def}}{=} \max_{1\le j\le n} \left(\frac{1}{\gamma_j b_j} \sum_{i=1}^{n} |a_{ij}|\gamma_i \right) < 1.$$

Now equation (1.3) will have an equilibrium point, $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$, if the u_i^* satisfy

$$b_i u_i = \sum_{j=1}^{n} a_{ij} g_j(u_j) + J_i, \quad i = 1, \ldots, n. \tag{3.2}$$

Multiplying equation i by $\gamma_i > 0$ yields the equivalent equations

$$\gamma_i b_i u_i = \sum_{j=1}^{n} \gamma_i a_{ij} g_j(u_j) + \gamma_i J_i, \quad i = 1, \ldots, n.$$

Define $v_i = \gamma_i b_i u_i$. Then the u_i satisfy (3.2) if and only if the v_i satisfy

$$\begin{aligned} v_i &= \sum_{j=1}^{n} \gamma_i a_{ij} g_j\left(\frac{v_j}{\gamma_j b_j}\right) + \gamma_i J_i, && (3.3) \\ &\overset{\text{def}}{=} G_i(v_1, v_2, \ldots, v_n). && (3.4) \end{aligned}$$

Thus we seek fixed points of the map $G : \mathrm{R}^n \to \mathrm{R}^n$, defined by $G = (G_1(v), \ldots, G_n(v))$, $v = (v_1, \ldots, v_n)$. Clearly, from the conditions (H1) and (H2) on the g_i, the G_i satisfy

$$\zeta_i^- = \gamma_i\left(J_i - \sum_{j=1}^n |a_{ij}|\right) \le G_i(v) \le \gamma_i\left(J_i + \sum_{j=1}^n |a_{ij}|\right) = \zeta_i^+.$$

Thus G maps the set

$$S = \{(v_1, \ldots, v_n) | \zeta_i^- \le v_i \le \zeta_i^+,\ i = 1, \ldots, n\}$$

into itself.

Further

$$\begin{aligned} \|G(v) - G(w)\| &= \sum_{i=1}^n |G_i(v) - G_i(w)| \\ &\le \sum_{i=1}^n \sum_{j=1}^n \gamma_i |a_{ij}| \left| f_j\left(\frac{v_j}{\gamma_j b_j}\right) - f_j\left(\frac{w_j}{\gamma_j b_j}\right)\right| \\ &\le \sum_{i=1}^n \sum_{j=1}^n \frac{\gamma_i |a_{ij}|}{\gamma_j b_j} |v_j - w_j| \end{aligned}$$

where the latter follows from condition (H1) and the mean value theorem. Rearranging the subscripts we see that

$$\begin{aligned} ||G(v) - G(w)|| &\le \sum_{j=1}^n \left(\frac{1}{\gamma_j b_j} \sum_{i=1}^n \gamma_i |a_{ij}|\right) |v_j - w_j| \\ &\le \beta \sum_{j=1}^n |v_j - w_j| \\ &= \beta ||v - w||, \end{aligned}$$

where $\beta < 1$. Thus G is a contraction on S and by the contraction mapping principle, G has a unique fixed point. It follows that (1.3) has a unique equilibrium point.

Theorem 3.3. Assume (H1) and (H2) hold. If $-\widehat{\mathcal{K}}$ is a nonsingular M-matrix then the equilibrium point of (1.3) is globally asymptotically stable.

Proof. Let $\gamma_i > 0$ be as in the previous proof and define

$$\mu = \min_{1\le j\le n}\left\{ b_j\gamma_j - \sum_{i=1}^{n} |a_{ij}|\gamma_i \right\}.$$

It follows from (3.1) that $\mu > 0$. Let $x_i(t) = u_i(t) - u_i^*$ and recall that the delay differential equation (2.1) governs the time evolution of $x_i(t)$. As mentioned in the introduction, we will assume that this equation is subject to continuous initial data of the form (1.4), in which case the local existence and uniqueness of solutions for $x_i(t)$ is guaranteed.

Consider a Lyapunov functional $V(t) = V(x)(t)$ defined by

$$V(x)(t) = \sum_{i=1}^{n} \gamma_i |x_i(t)| + \sum_{i=1}^{n} \gamma_i \sum_{j=1}^{n} |a_{ij}| \int_{t-\tau_{ij}}^{t} |x_j(s)|\, ds. \tag{3.5}$$

Then the upper righthand derivative of V along the solutions of (1.3) is given by

$$\begin{aligned}
D^+V(t) \;&\le\; \sum_{i=1}^{n} \gamma_i \Big(-b_i|x_i(t)| + \sum_{j=1}^{n} |a_{ij}||g_j(x_j(t-\tau_{ij}) + u_j^*) - g_j(u_j^*)| \Big) \\
&\quad + \sum_{i=1}^{n} \gamma_i \sum_{j=1}^{n} |a_{ij}| \left(|x_j(t)| - |x_j(t-\tau_{ij})| \right) \\
&\le\; \sum_{i=1}^{n} (-\gamma_i b_i |x_i(t)|) + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i |a_{ij}||x(t-\tau_{ij})| \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i |a_{ij}||x_j(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i |a_{ij}||x_j(t-\tau_{ij})| \\
&\le\; \sum_{j=1}^{n} \Big(-\gamma_j b_j + \sum_{i=1}^{n} \gamma_i |a_{ij}| \Big) |x_j(t)| \\
&\le\; -\mu \sum_{j=1}^{n} |x_j(t)|.
\end{aligned}$$

The boundedness of $\sum_{i=1}^{n} |x_i(t)|$ for all $t \ge 0$ follows from (3.5) and (3.6). Thus, the solutions of (2.1) exist for all $t \ge 0$. We have from (3.5) and (3.6) that

$$V(x)(t) + \mu \int_0^t \left(\sum_{i=1}^{n} |x_i(s)| \right) ds \le V(x)(0).$$

It follows that

$$\sum_{i=1}^{n} |x_i(t)| \in \mathrm{L}_1(0,\infty). \tag{3.6}$$

The boundedness of $x_i(t)$ on $(0,\infty)$ implies that of the derivative of $x_i(t)$ on $(0,\infty)$; hence $x_i(t)$ is uniformly continuous on $(0,\infty)$. The uniform continuity of $\sum_{i=1}^{n} |x_i(t)|$ on $(0,\infty)$ together with (3.6) implies that

$$\sum_{i=1}^{n} |x_i(t)| \to 0 \qquad \text{as } t \to \infty. \tag{3.7}$$

This last assertion is a consequence of a lemma due to Barbălat [3]. (The statement and proof of this lemma can be found in the book of Gopalsamy [12, pp. 4–5]).

It was pointed out in [41] that [14, Proposition 1] remains valid (with a slight modification of the Lyapunov function) under weaker assumptions on the nonlinearities: each of the g_j is globally Lipschitz and globally bounded. The same is true of Theorem 3.3.

4. Comparison of Results

Campbell [8] considered a delayed additive neural network which is a unidirectional ring, and is modelled by the following system:

$$\dot{u}_i(t) = -b_i u_i(t) + a_{ii} g_i(u_i(t-\tau_{ii})) + a_{i,i-1} g_{i-1}(u_{i-1}(t-\tau_{i-1})), \tag{4.1}$$

$i = 1, \ldots, n$. Due to the structure of system, the characteristic equation is amenable to direct analysis and the exact region of linearized stability of the trivial solution of (4.1) can be described. In this section, we will use this model and its stability region as a benchmark for the criteria for local and global stability obtained in Sections 2 and 3 and in the work of other authors. That is, we will compare the regions of stability predicted by each set of criteria when applied to this model.

4.1. Linear Stability

We begin our comparison of criteria for linearized stability by stating a slight generalization of a result of [8] concerning stability independent of all the delays.

Theorem 4.1. The trivial solution of (4.1) is linearly asymptotically stable for all $\tau_{ij} \geq 0$ if

$$|a_{ii}| < b_i,\ i = 1, \ldots n \quad \text{and} \quad \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n}(b_i - |a_{ii}|). \tag{4.2}$$

If $|a_{ii}| > b_i$ for at least one i, then there no such region of delay independent stability.

The Theorem in [8] is stated for the case that the diagonal delays are all equal, i.e., $\tau_{ii} = \tau_s,\ i = 1, \ldots, n$. However, the proof there holds in the case that the τ_{ii} are not equal.

Consideration of the trivial solution of model (4.1) shows that for this model the matrix $\mathcal{K} = K$ and is given by

$$\begin{bmatrix} -b_1 + |a_{11}| & 0 & \cdots & 0 & |a_{1n}| \\ |a_{21}| & -b_2 + |a_{22}| & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & |a_{n,n-1}| & -b_n + |a_{nn}| \end{bmatrix} \tag{4.3}$$

Applying Corollary 2.7 shows that the trivial solution of (4.1) will be linearly asymptotically stable for all $\tau_{ij} > 0$ if

$$-b_i \leq a_{ii} < b_i, \quad i = 1, \ldots n$$

and

$$\begin{aligned} \prod_{i=1}^{n}(b_i - a_{ii}) \leq \prod_{i=1}^{n} a_{i,i-1} < \prod_{i=1}^{n}(b_i - a_{ii}),\ &\text{if } a_{ii} > 0,\ i = 1, \ldots, n, \\ \left|\prod_{i=1}^{n} a_{i,i-1}\right| \leq \prod_{i=1}^{n}(b_i - |a_{ii}|),\ &\text{otherwise.} \end{aligned} \tag{4.4}$$

There will be no delay independent stability if $|a_{ii}| > b_i$ for at least one i or

$$\left|\prod_{i=1}^{n} a_{i,i-1}\right| > \prod_{i=1}^{n}(b_i - |a_{ii}|).$$

This gives a slight extension to Theorem 4.1.

The following is a restatement and generalization of a result of [8] concerning stability independent of the *off-diagonal* delays.

Theorem 4.2. The trivial solution of (4.1) is linearly asymptotically stable for all $\tau_{ij} > 0,\ i \neq j$ if

$$\begin{aligned} a_{ii} &< b_i,\ i = 1, \ldots n; \\ \tau_{ii} &< \frac{1}{b_i}\left(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1\right), i \quad \text{such that } a_{ii} < 0; \end{aligned} \tag{4.5}$$

and

$$\left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n} (b_i - a_{ii}). \tag{4.6}$$

Proof. Let $\beta \overset{\text{def}}{=} \prod_{i=1}^{n} a_{i,i-1}$ and $\alpha \overset{\text{def}}{=} \prod_{i=1}^{n} (b_i - a_{ii})$. Note that if $-b_i < a_{ii} < b_i$ or $a_{ii} < -b_i$ and $\tau_{ii} < \frac{1}{\sqrt{a_{ii}^2 - b_i^2}} \text{Arccos}(\frac{b_i}{a_{ii}}) < \frac{1}{b_i}(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1)$, then the trivial solution will be stable for $\beta^- < \beta < \min\{\beta^+,\ \alpha\}$, where $\beta = \beta^\pm$ correspond to curves along which the characteristic equation of (4.1) has a pair of pure imaginary roots.

It is shown in [8] that

$$\begin{aligned} \beta^{\pm 2} &= \prod_{i=1}^{n} [(b_i - a_{ii} \cos \omega\tau_{ii})^2 + (\omega + a_{ii} \sin \omega\tau_{ii})^2] \\ &\overset{\text{def}}{=} \prod_{i=1}^{n} \beta_i^2(\omega) \end{aligned}$$

Now if $a_{ii} > 0$ then $(b_i - a_{ii} \cos \omega\tau_{ii})^2 \geq (b_i - a_{ii})^2$ and $(\omega + a_{ii} \sin \omega\tau_{ii})^2 \geq 0$ thus $\beta_i^2(\omega) \geq \beta_i^2(0)$, with equality only when $\omega = 0$. Further, by [8, Theorem 2.5], if $\tau_{ii} < \frac{1}{b_i}(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1)$ then $\beta_i'(\omega) \geq 0$ with equality at $\omega = 0$ implying $\beta_i^2(\omega) \geq \beta_i^2(0)$. Thus if this condition is satisfied for each i such that $a_{ii} < 0$ then we have $\beta^{\pm 2} \geq \beta^{\pm}(0) = \alpha^2$ and the result follows.

Note that in the case $b_i = 0, i = 1, \ldots, n$ conditions (4.5)–(4.6) become

$$a_{ii} < 0, \quad -a_{ii}\tau_{ii} < \frac{1}{2}, \quad \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n} (-a_{ii}), \qquad i = 1, \ldots, n. \tag{4.7}$$

Recently, So, Tang and Zou [38] obtained the following result for general networks of the form (1.3), with $b_i = 0$.

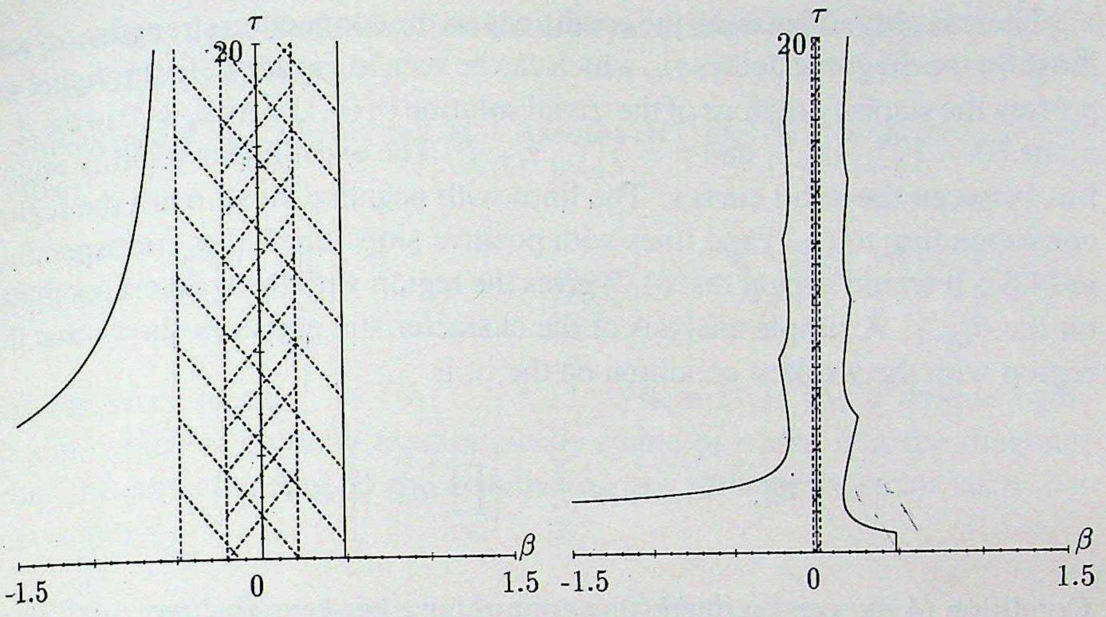

Figure 1. Regions of stability in the β, τ plane for the three element ring network (4.1) with $b_i = 0, a_{11} = -0.5, a_{22} = -0.8, a_{33} = -1.2$. (a) $\tau_{ii} = 0.4$ (b) $\tau_{ii} = 1.0$. The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines with negative slope indicate the region of off-diagonal delay independent linear stability as predicted by Theorem 4.2 [8]. Dashed lines with positive slope indicate the region as predicted by Theorem 4.3 [38].

Theorem 4.3 [38, Theorem 1.2]. Let $\widetilde{C}$ be the matrix with elements $\tilde{c}_{ii} = -c_{ii}$,

$$\tilde{c}_{ij} = -\frac{1 - \frac{1}{9}c_{ii}\tau_{ii}(3 - 2c_{ii}\tau_{ii})}{1 + \frac{1}{9}c_{ii}\tau_{ii}(3 - 2c_{ii}\tau_{ii})}|c_{ij}|, \; i \neq j,$$

and assume that $-c_{ii}\tau_{ii} < \frac{3}{2}, i = 1, \ldots, n$. If $\widetilde{C}$ is a non-singular M-matrix, then every solution $(x_1(t), x_2(t), \ldots, x_n(t))$ of (2.2) tends to 0 as $t \to \infty$.

In the case of the ring network (4.1) these conditions reduce to

$$a_{ii} < 0, -a_{ii}\tau_{ii} < \tfrac{3}{2}, \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n}(-a_{ii})\frac{1 + \frac{1}{9}a_{ii}\tau_{ii}(3 - 2a_{ii}\tau_{ii})}{1 - \frac{1}{9}a_{ii}\tau_{ii}(3 - 2a_{ii}\tau_{ii})}, \quad i = 1, \ldots, n. \tag{4.8}$$

There is playoff between the conditions on the connection strengths, a_{ij} and those on the diagonal delays τ_{ii}, which can be seen in Figure 1. In this figure we portray the stability regions of the trivial solution of (4.1) with $b_i = 0$ in the β, τ plane ($\beta = \prod_{i=1}^{n} a_{i,i-1}$ and $\tau = \sum_{i=1}^{n} \tau_{i,i-1}$). The exact linear stability region lies between the solid curves. The lines with negative slope mark the region corresponding to (4.7) and lines with positive slope mark that corresponding to (4.8). It would appear that (4.7) gives the region with the weakest condition on the $a_{i,i-1}$. A simple analysis of the characteristic equation shows that the region with the weakest condition on the τ_{ii} is

$$a_{ii} < 0, \quad \tau_{ii} < \frac{\pi}{2}, \quad \left|\prod_{i=1}^{n} a_{i,i-1}\right| = 0.$$

Condition (4.8) gives an interesting compromise between the two.

4.2. Global Stability

Now consider the result in Section 3. When applied to the ring network (4.1), Theorem 3.3 predicts global stability exactly in the region given by (4.2). In [14], Gopalsamy and He obtained the following sufficient condition for global stability independent of the delays:

$$\max_{1\le i\le n}\left\{\frac{1}{b_i}\sum_{j=1}^{n}|a_{ji}|\right\}. < 1 \tag{4.9}$$

which for system (4.1) can be written as

$$|a_{i,i-1}| < b_i - |a_{ii}|, 1 \le i \le n. \tag{4.10}$$

Clearly conditions (4.10) imply conditions (4.2).

Since (4.2) is close to being sharp, in the sense that the condition for delay independent local stability is almost the same, it seems likely that to get any weaker conditions on the a_{ij} which guarantee global stability, one will need to have conditions involving the delays. For example, recent work of van den Driessche, Wu and Zou [40, 41] has obtained conditions for global stability in the network (1.3) (in the case $b_i = 1$) which depend on the *diagonal* delays.

This work uses the techniques of monotone dynamical systems theory to obtain the following result:

Theorem 4.4 [40, Theorem 2.1]. Assume (H1) and (H2) hold, and $\tau_{ij} > 0$. If either

$$\max_{1\le i\le n}\left\{a_{ii} + \sum_{j\ne i}^{n} |a_{ij}|\right\} < 1 \qquad \text{or} \qquad \max_{1\le i\le n}\left\{a_{ii} + \sum_{j\ne i}^{n} \frac{|a_{ij}| + |a_{ji}|}{2}\right\} < 1,$$

then for every input $J = (J_1, \dots, J_n)^T$, system (1.3) has a unique equilibrium, which is globally asymptotically stable provided that the diagonal delays τ_{ii} corresponding to the negative a_{ii} are sufficiently small such that $0 \le \tau_{ii} \le \frac{1}{1-ea_{ii}}$.

If these conditions are applied to system (4.1), they become

$$|a_{i,i-1}| < 1 - a_{ii}, \qquad 1 \le i \le n. \tag{4.11}$$

and

$$0 \le \tau_{ii} \le \frac{1}{1 - ea_{ii}}, \tag{4.12}$$

for each i such that $a_{ii} < 0$. Clearly condition (4.11) is weaker than either of the conditions (4.2), (4.10), except in the case $a_{ii} \ge 0,\ i = 1, \dots, n$ in which case (4.10) and (4.11) are equivalent. This is not surprising; van den Driessche et al. [40] have been able to weaken the condition on the a_{ii} by adding a condition on the τ_{ii}. Note further that both (4.11),(4.12) are stronger than the analogous ones obtained for off-diagonal delay independent local stability in [8], i.e (4.5), (4.6).

We illustrate these results in the β, τ plane in Figures 2 and 3 ($\beta = \prod_{i=1}^{n} a_{i,i-1}$ and $\tau = \sum_{i=1}^{n} \tau_{i,i-1}$). In this parameter space the result of [14] appears the same as the result of Theorem 3.3.

5. Conclusions

We have given necessary and sufficient conditions for delay independent, asymptotic stability of the trivial solution of the linear delay differential equation (2.2) and shown how they can be modified to give necessary and sufficient conditions for the local asymptotic stability of equilibrium points of the general

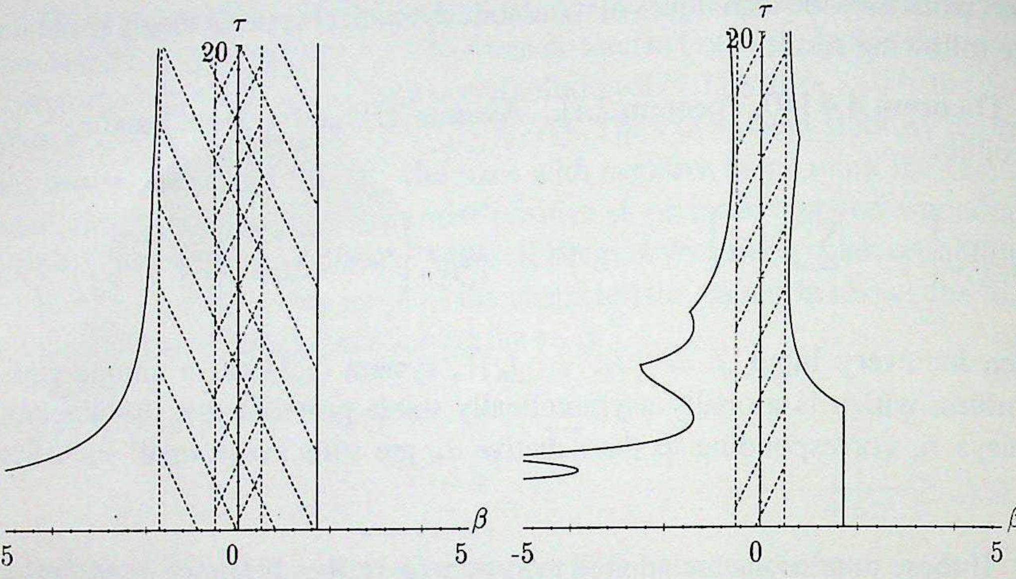

Figure 2. Region of stability in the β, τ plane for the three element ring network (4.1) with $b_i = 1, a_{11} = -0.1, a_{22} = -0.2, a_{33} = -0.3$. (a) $\tau_{ii} = 0.5$ (b) $\tau_{ii} = 10$. The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines with negative slope indicate the region of global stability as predicted by Theorem 4.4 [40]. Dashed lines with positive slope indicate the region of global stability as predicted by Theorem 3.3.

additive neural network (1.3). We further showed that a slightly strengthening of these conditions yields sufficient conditions for the nonlinear system to possess a unique equilibrium solution, and for this solution to be global asymptotically stable independent of the delays. Our linear results are slightly stronger than those of [24] as we allow delays in the self feedback terms. They are complementary to those of [38] who impose weaker conditions on the connection matrix, but stronger conditions on the diagonal delays. Our global results are stronger to those of [14] and complementary to those of [40], as described above.

The stability results derived and reviewed in this article all contain the following:

(A1) conditions on some or all of the "diagonal parameters" b_i, a_{ii}, and τ_{ii};

(A2) a condition forcing the off-diagonal connection coefficients, a_{ij}, to be

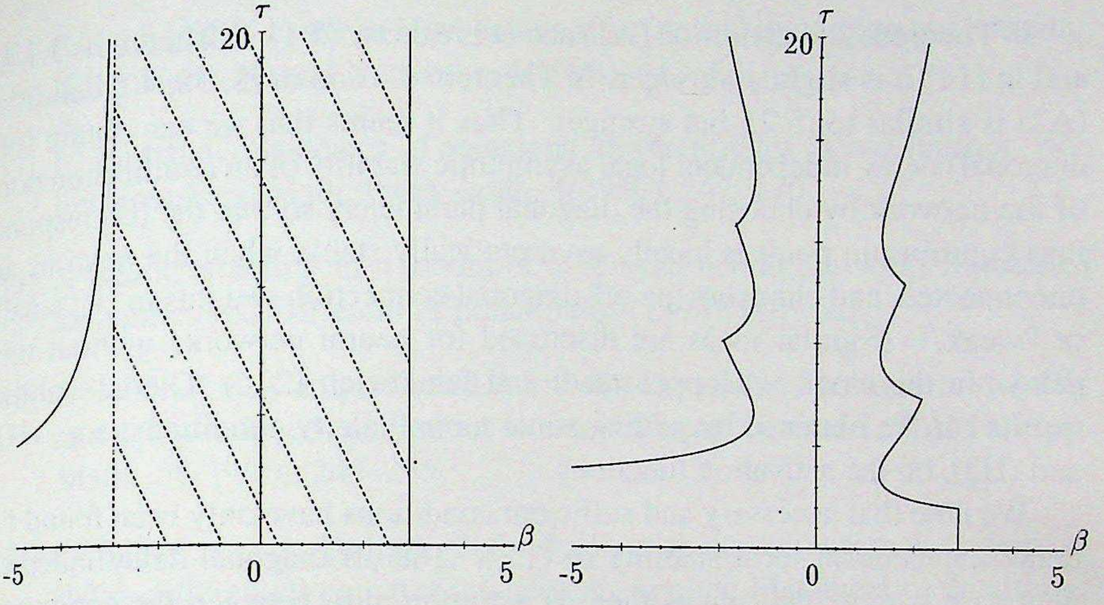

Figure 3. Region of stability in the β, τ plane for the three element ring network (4.1) with $b_i = 1, a_{11} = -0.1, a_{22} = -0.2, a_{33} = -1.2$. (a) $\tau_{ii} = 0.2$ (b) $\tau_{ii} = 1.5$. The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines indicate the region of global stability as predicted by [40].

"small" in comparison with the diagonal parameters.

When these conditions are independent of τ_{ii}, as in Theorems 2.6, 3.3 and 4.1 and in [14], [24] one obtains what is usually called *delay independent stability*. When they depend on the τ_{ii}, as in Theorem 4.2 and in [8, 38, 40], one obtains what can be described as *off-diagonal delay independent stability*.

To see why such conditions might lead to stability, we return to the simple case of a group of unconnected elements, i.e., (1.3) with $a_{ij} = 0, i \neq j$. In this situation it is well known [28, 39] that equilibrium points u^* of the system will be locally asymptotically stable for

$$-b_i \leq a_{ii} < b_i, \qquad \tau_{ii} \geq 0 \tag{5.1}$$

and for

$$a_{ii} < -b_i, \tau_{ii} < \frac{1}{\sqrt{a_{ii}^2 - b_i^2}} \operatorname{Arccos} \frac{b_i}{a_{ii}}. \tag{5.2}$$

In Theorem 2.6, condition (A1) above is exactly (5.1); in Theorems 3.3,4.1 and in [14] it is slightly stronger. In Theorem 4.2 and in [8, 38, 40] condition (A2) is similar to (5.2), but stronger. Thus it seems that we can obtain (off-diagonal) delay independent local asymptotic stability of an equilibrium point of the network by choosing the diagonal parameters so that the (corresponding) equilibrium point is locally asymptotically stable when the neurons are unconnected, and choosing the off-diagonal connection weights to be "small" or "weak." (Similar ideas are discussed for neural networks without time delays in the work of Hoppensteadt and Izhikevich [27].) Global stability results can be obtained by adding some monotonicity conditions, e.g., (H1) and (H2), on the activation functions.

We note that necessary and sufficient conditions have only been found for delay independent local stability in (1.3). The off-diagonal delay independent case is more delicate as there is a compromise between the conditions imposed on the diagonal parameters and those on the off-diagonal parameters. As might be expected, one can find weaker conditions on one set of parameters by imposing stronger conditions on the other set. For Lotka-Volterra systems it has been shown [24, 36] that delay independent conditions such as those discussed in this article are actually necessary and sufficient for delay independent *global* stability. To our knowledge, no such results have yet been obtained for the model (1.3).

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Chaotic Dynamics of Neural Networks and its Application to Combinatorial Optimization

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Abstract

This article reviews our analyses on chaotic dynamics of neural networks and its application to combinatorial optimization. We first provide sufficient conditions for the asymptotical stability and local bifurcations of transiently chaotic neural networks (TCNNs), which are a generalized version of discretized Amari-Hopfield neural networks, and then show that there exists a topologically chaotic structure with a snap-back repeller in a global attracting set. Furthermore, when certain conditions are satisfied, the attracting set eventually evolves into a strange attractor by a homoclinic bifurcation, which results in complicated chaotic dynamics. The existence of the strange attractor implies that TCNNs have a globally searching ability for globally optimal solutions of the commonly used objective functions in combinatorial optimization problems when the size of the strange attractor is sufficiently large. In addition, since chaos may be viewed as deterministic noise, we develop a deterministic optimization method, which is called as chaotic simulated annealing. Finally, TCNNs are applied to the combinatorial problems for demonstrating their globally searching ability and efficiency of the chaotic simulated annealing.

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1. Introduction

Both theoretical and experimental studies have indicated that nonlinear dynamics and chaos may play important roles in neural information processing of the brain. Understanding dynamical structure and chaos has important implications not only for elucidating the nonlinearity, diversity and even capabilities of neural networks, but also for providing fundamental ideas to a wider range of applications in neural computing. This article aims to review our recent works on neural networks with special emphasis on chaotic dynamics for both theoretical studies and applications [4], [6]-[11].

In the theoretical works, we analyse both local and global structures of the transiently chaotic neural networks (TCNNs), which are a generalized version of the discretized Amari-Hopfield neural networks [3, 21], and give sufficient conditions for asymptotical stability, local bifurcations and chaos. The relations between the stable fixed points and local minima of the commonly used objective functions are also investigated. In particular, we show that there exists a topologically chaotic structure in TCNNs, which eventually evolves into a strange attractor by global homoclinic bifurcation, when certain conditions are satisfied. The strange attractor is also the global unstable set of a repeller, which is asymptotically stable, sensitively dependent on initial conditions and topologically transitive. Since this attractor can cover all global minima for the objective function, we show that TCNNs have global searching ability that enables the network to escape from local minima and search for globally optimal solutions of optimization problems.

On the other hand, when the absolute value of self-feedback is slowly reduced with time evolution, dynamical process of TCNNs is similar to the annealing mechanism, i.e., from irregular chaotic dynamics through successive bifurcations finally to a stable equilibrium. Therefore, in the application works, we first establish a deterministically global optimization algorithm, i.e., chaotic simulated annealing (CSA), which views chaos as deterministic noise, in contrast to the stochastic noise in the conventional stochastic simulated annealing (SSA). Then TCNNs are applied to a well-known combinatorial optimization problem, or the traveling salesman problem (TSP), by exploiting the globally searching ability. Besides TSP, many experiment works have also verified that CSA has very high ability to reach global minima with remarkable efficiency for various optimization problems [23, 24, 26, 33, 36, 37, 38, 39].

Based on our recent works [7, 8, 10, 11], this article is organized as follows: In the next section, several definitions related to an attracting set and a strange attractor are briefly described. In Section 3, we first give a general form of chaotic neural networks (CNNs) and TCNNs and then show that TCNNs are a generalized version of the discretized Amari-Hopfield neural networks [3, 20]. Section 4 is local analysis, where the conditions of asymptotical stability and local bifurcations are provided. On the other hand, Section 5 is global analysis for topological structure, where we show that there exists a global attracting set with a snap-back repeller, i.e., topological chaos. In Section 6, the attracting set is further proven to be a strange attractor whose size can be sufficiently large such that all minima of the objective function are involved, when certain conditions are satisfied. The chaotic simulated annealing is explained in Section 7, and an example of the application of TCNNs to optimization problems are given in Section 8. Finally we make several general remarks to conclude this article in Section 9.

2. Basic Definitions and Notations

We introduce some concepts and definitions in this section, which are used throughout the article. Define a discrete-time system as

$$Y(t+1) = F(Y(t)), \tag{1}$$

where $Y(t) \in \mathcal{R}^n$, $(t = 1, 2, \ldots)$ and $F \in C^1 : \mathcal{R}^n \mapsto \mathcal{R}^n$.

Let F^m denote the composition of F with itself m times. A point Y is a p-periodic point if $F^p(Y) = Y$ but $F^m(Y) \neq Y$ for $1 \leq m < p$. If $p = 1$, that is, $F(Y) = Y$, Y is called a fixed point. Let $F_Y(Y)$ and $det(F_Y(Y))$ be the Jacobian matrix of F at a point $Y \in \mathcal{R}^n$ and its determinant, respectively. Let $B(Y;\alpha)$ be a closed ball in $\mathcal{R}^n$ of radius $\alpha (> 0)$ centered at the point Y. Y^T is the transpose of a column vector Y. $\|\cdot\|$ is the usual Euclidean norm of a vector or the induced matrix norm of a matrix. $\emptyset$ stands for empty set. 0 is the zero matrix with appropriate dimensions. $\mathcal{O}(y)$ means that this term is in the order of y. A neighbourhood U of a point Y or a set V is an appropriate open set including Y or V, respectively. Define $E_0 = (0, \ldots, 0)^T, E_1 = (1, \ldots, 1)^T$, and E to be the identity matrix .

Define the region $S = \{(x_1, \ldots, x_n) | 0 \leq x_i \leq 1; i = 1, \ldots, n\}$. Then S can be divided into vertices $S_V = \{(x_1, \ldots, x_n) | x_i = 0 \text{ or } 1; i = 1, \ldots, n\}$, internal points $S_I = \{(x_1, \ldots, x_n)|0 < x_i < 1; i = 1, \ldots, n\}$ and boundary points $S_B = S - S_I - S_V$, respectively. $x \propto y$ means that x is directly proportional to y. Unless specially defined, a small letter stands for a scalar, and a capital letter represents a vector, a matrix or a set in this article.

The locally stable and unstable sets (or manifolds) of a hyperbolic fixed point $\overline{Y}$ are defined for a neighbourhood U of $\overline{Y}$ where $F(Y)$ is locally invertible for $Y \in U$, as follows:

$$W^s_{loc}(\overline{Y}) = \{Y \in U|F^m(Y) \to \overline{Y} \text{ as } m \to +\infty, \text{ and } F^m(Y) \in U, \forall m \geq 0\},$$
$$W^u_{loc}(\overline{Y}) = \{Y \in U|F^{-m}(Y) \to \overline{Y} \text{as} m \to +\infty, \text{and} F^{-m}(Y) \in U, \forall m \geq 0\}.$$

Then, we have the definitions of the global stable and unstable sets as follows

$$\begin{aligned} W^s(\overline{Y}) &= \{Y \in \mathcal{R}^n|F^m(Y) \to \overline{Y} \text{as} m \to +\infty\}, \\ W^u(\overline{Y}) &= \bigcup_{m \geq 0} F^m(W^u_{loc}(\overline{Y})). \end{aligned}$$

An invariant set Λ is a subset $\Lambda \subset \mathcal{R}^n$ such that $F(\Lambda) = \Lambda$. Next we summarize necessary definitions [5, 16, 17, 29] used in this article.

Definition 1. A closed invariant set Λ is called an attracting set if there is some neighborhood V of Λ such that $F(V) \subset V$ and $F^m(Y) \to \Lambda$ as $m \to \infty$, for all $Y \in V$. The set $U = \{Y \in \mathcal{R}^n|F^m(Y) \to \Lambda \text{ as } m \to +\infty\}$ is the domain (or basin) of attraction of Λ, which is an open set.

We use Lyapunov stability to describe the dynamical stability for an invariant set. A closed invariant set Λ is Lyapunov stable if, for every open neighbourhood U of Λ, there exists an open neighbourhood V of Λ such that $F^m(V) \subset U$ for all non-negative m. By the definition of the Lyapunov stability, we have the definition of asymptotical stability.

Definition 2. A closed invariant set Λ is said to be asymptotically stable if it is Lyapunov stable and there is an open neighbourhood W of Λ such that $\omega(Y) \subset \Lambda$ for all $Y \in W$.

Moreover, describing a strange attractor requires the definitions of transitivity, transversal homoclinic intersection and sensitivity on initial values.

Definition 3. A closed invariant set Λ is topologically transitive if for any two open sets $U, V \in \Lambda$, a positive integer m exists such that $F^m(U) \cap V \neq \emptyset$, or equivalently a point $Y \in \Lambda$ exists such that its orbit is dense in Λ.

If a set Λ is a topologically transitive set, Λ is indecomposable.

Definition 4. An orbit starting at a point $Y(\neq \overline{Y}) \in \mathcal{R}^n$ is called homoclinic to the fixed point $\overline{Y}$ if $F^m(Y) \to \overline{Y}$ as $m \to \pm\infty$. Moreover, if $W^s(\overline{Y})$ and $W^u(\overline{Y})$ intersect transversally at Y, the orbit is called a transversal homoclinic orbit.

Next, we give a definition for sensitive dependence on initial conditions.

Definition 5. F is sensitively dependent on initial conditions on an invariant set $\Lambda \subset \mathcal{R}^n$ if there exists $\delta > 0$ such that, for any $Y \in \Lambda$ and any neighborhood V of Y, there exists $X \in V$ and $m > 0$ such that $\|F^m(Y) - F^m(X)\| > \delta$.

Then we have the definitions of the attractor and the strange attractor [17, 16, 29].

Definition 6. An attractor is an attracting set which is topologically transitive. An attractor is strange if it contains a transversal homoclinic orbit and is sensitively dependent on initial conditions.

It is evident that an attractor in Definition 6 is both asymptotically stable and indecomposable. Therefore, attractors defined in this article are asymptotically stable attractors. They are sometimes referred to as open-basin attractors although this terminology may be misleading [5]. To describe the topological properties of chaos based on Marotto Theorem[28], next we also give the definitions of a snap-back repeller and formal chaos.

Definition 7. A fixed point $\overline{Y}$ is said to be a snap-back repeller of Eqn. (1) if there exists a real number $\alpha(> 0)$ and a snap-back point $Y^0 \in B(\overline{Y}; \alpha)$ with $Y^0 \neq \overline{Y}$, such that all eigenvalues of $F_Y(Y)$ exceed unity in norm for all $Y \in B(\overline{Y}; \alpha)$ and $F^m(Y^0) = \overline{Y}$ with $det(F_Y^m(Y^0)) \neq 0$ for some positive integer m.

Theorem 1. If Eqn. (1) has a snap-back repeller, then the system of Eqn. (1) is chaotic in the sense of Marotto. In other words, there exist,

(A) a positive integer m_0 such that for each integer $p \geq m_0$, Eqn. (1) has $p-$periodic points;

(B) a scrambled set L, i.e, an uncountable set containing no periodic points such that

  (a) $F(L) \subset L$,

  (b) for every $Y \in L$ and any periodic point X of Eqn. (1),

$$\limsup_{m\to\infty} \|F^m(Y) - F^m(X)\| > 0,$$

  (c) for every $X, Y \in L$ with $X \neq Y$,

$$\limsup_{m\to\infty} \|F^m(Y) - F^m(X)\| > 0,$$

(C) an uncountable subset L_0 of L such that for every $X, Y \in L_0$,

$$\liminf_{m\to\infty} \|F^m(Y) - F^m(X)\| = 0.$$

Strange attractors in Definition 6 emphasize an attracting property for chaos. In contrast, Marotto's theorem describes chaos from the topological viewpoint, which is formal chaos and may not be attracting. Actually the Marotto chaos is a generalization of the Li-Yorke chaos in a one-dimensional difference equation [27] to multi-dimensional systems.

3. Transiently Chaotic Neural Networks

Transiently chaotic neural networks (TCNNs) can be described in terms of scalar variables as following difference equations [6, 7, 8, 10, 11]:

$$y_i(t+1) = ky_i(t) + \sum_{j=1}^{n} \omega_{ij} x_j(t) + a_i - \omega_{ii} a_{0i} \qquad (2)$$

$$x_i(t) = \frac{1}{1 + e^{-y_i(t)/\epsilon}} \qquad (i = 1, \ldots, n) \qquad (3)$$

where all variables and parameters are real numbers and

x_i : output of neuron i
y_i : internal state of neuron i
ω_{ij} : connection weight from neuron j to neuron i
a_i : input bias of neuron i
k : damping factor of nerve membrane
ϵ : steepness parameter of the output function ($\epsilon > 0$)
ω_{ii} : self-recurrent connection weight of neuron i
a_{0i} : self-recurrent bias of neuron i.

Originally ω_{ii} is a function of t in TCNNs[7, 8, 10], whose absolute value is slowly reduced during time evolution. For the sake of the simplification of description, we assume that all the self-recurrent connection weights ω_{ii} are fixed and considered as parameters in the theoretical analysis of the subsequent sections except Sections 7 and 8. For the applications of TCNNs in Sections 7 and 8, $|\omega_{ii}|$ are taken as variables and gradually reduced to emulate the annealing process.

Therefore, TCNNs can simply be rewritten in terms of vector variables as follows.

$$Y(t+1) = F(Y(t)) = kY(t) + WX(t) + I - I_0 \tag{4}$$

where $F(Y(t)) = [f_1(Y(t)), \ldots, f_n(Y(t))]^T$, $Y(t) = [y_1(t), \ldots, y_n(t)]^T$, $X(t) = [x_1(t), \ldots, x_n(t)]^T$,

$$I = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad I_0 = \begin{bmatrix} \omega_{11}a_{01} \\ \vdots \\ \omega_{nn}a_{0n} \end{bmatrix}, \quad W = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{bmatrix}.$$

Clearly, F in Eqn. (4) is a C^1 class function. If $I_0 = 0$ and ω_{ii} are constants, Eqn. (4) coincides with the original chaotic neural networks (CNNs) [4, 7, 8]. For more particular cases, if $\omega_{ii} = 0$ for all i, TCNNs reduce to the difference equational version[7] of the Amari-Hopfield neural networks[3, 21, 22] on the basis of the Euler method; if $k = 0$, $I_0 = 0$ and ω_{ii} are constants, TCNNs are equivalent to the mean field equations (mean field Ising model) [9, 30].

TCNNs generate rich spatio-temporal dynamics with various coexisting attractors, not only of fixed points but also of periodic and even more complex attractors[8, 10, 11, 34, 35]. Numerical computations have verified that TCNNs have a very high ability to find globally optimal solutions for combinatorial problems [7, 23, 24, 26, 33, 36, 37, 38, 39] although they are deterministic models without stochastic fluctuations.

To analyse the asymptotical stability of $X(t)$, we rewrite Eqn. (4) by explicitly expressing with $X(t)$,

$$X(t+1) = \overline{F}(X(t)) = \begin{bmatrix} \dfrac{1}{1+e^{-\frac{1}{\epsilon}[k\epsilon ln \frac{x_1(t)}{1-x_1(t)} + \sum_{j=1}^{n} \omega_{1j} x_j(t) + a_1 - \omega_{11} a_{01}]}} \\ \vdots \\ \dfrac{1}{1+e^{-\frac{1}{\epsilon}[k\epsilon ln \frac{x_n(t)}{1-x_n(t)} + \sum_{j=1}^{n} \omega_{nj} x_j(t) + a_n - \omega_{nn} a_{0n}]}} \end{bmatrix} \tag{5}$$

where $0 \leq x_i(t) \leq 1$ for $i = 1, \ldots, n$.

4. Local Structure and Bifurcations

In this section, we first describe several convergence theorems for TCNNs, and then derive the relations between asymptotically stable fixed points of TCNNs and local minimal points of a computational energy function which is usually used as an objective function in the optimization problems. In the last subsection, the local bifurcations and their effects to the minimal points are investigated. All the proofs of the theorems in this section are given in [8].

4.1. Asymptotical Stability at Fixed Points

We use a Lyapunov function to show several convergent theorems for Eqn. (5), which ensure that the system of Eqn. (5) asymptotically converges to a stable fixed point. Let us define two updating schemes, i.e., synchronous updating and asynchronous updating. In synchronous updating, all neurons are updated in parallel, using only old values as input. In asynchronous updating, neurons are updated one by one, using fresh values of previously updated neurons [13]. Generally, asynchronous updating can further be divided into random-order asynchronous updating where neurons are chosen in random order, and fixed-order asynchronous updating where neurons are chosen in fixed-order, e.g., cyclic updating. In this article, asynchronously updating means both random-order and fixed-order asynchronous updating.

Let $\lambda_{\min}, \lambda_{\max}$ be the smallest eigenvalue and the largest eigenvalue of the matrix W, respectively. Define a computational energy function $f(X)$ as follows:

$$f(X)=-\frac{1}{2}\sum_{ij}^{n}\omega_{ij}x_i x_j+\sum_{i}^{n}(a_{ii}-a_{0i}\omega_{ii})x_i+(1-k)\epsilon\sum_{i}^{n}\int_0^{x_i} ln\frac{x}{1-x}dx \tag{6}$$

which is generally used as an objective function for combinatorial optimization [7, 8].

Note that $\sum_i^n \int_0^{x_i} ln\frac{x}{1-x}dx = \sum_i^n [x_i ln x_i + (1-x_i)ln(1-x_i)]$ is bounded, thereby $f(X):\ \boldsymbol{R}^n \rightarrow\ \boldsymbol{R}$ is also bounded for all $0 \le x_i \le 1 (i = 1, \ldots, n)$. When $f(X)$ is used as an objective function in optimization problems, its last term is usually omitted due to small value of $(1-k)\epsilon$.

Next, we derive two theorems by proving that $f(X)$ is actually a Lyapunov function for the neural networks for synchronous and asynchronous updating, respectively.

Theorem 2. Assume $W = W^T$, and

1. $1/3 \ge k \ge 0, 4(1-k)\epsilon > -\lambda_{\min}$ or
2. $1 \ge k \ge 1/3, 8k\epsilon > -\lambda_{\min}$ or
3. $k > 1, 8\epsilon > -\lambda_{\min}$.

Then $X(t)$ of Eqn. (5) asymptotically converges to a fixed point by synchronous updating.

Theorem 3. Assume $W = W^T$, and

1. $1/3 \ge k \ge 0, 4(1-k)\epsilon > -\min_{i=1}^{n} \omega_{ii}$ or
2. $1 \ge k \ge 1/3, 8k\epsilon > -\min_{i=1}^{n} \omega_{ii}$ or
3. $k > 1, 8\epsilon > -\min_{i=1}^{n} \omega_{ii}$.

Then $X(t)$ of Eqn. (5) asymptotically converges to a fixed point by asynchronous updating.

Actually the attractors of Eqn. (5) for both synchronously and asynchronously updating, are only fixed points and 2-periodic points regardless of any value of ω_{ij} $(i, j = 1, \ldots, n)$ [8] if $W = W^T$ and $k = 0$. Although Theorems 2 and 3 are simple and easy to apply, they are conservative for the evaluation of asymptotical stability. Next, we give two more general theorems to evaluate the local stability.

For synchronous updating, we define a generalized Jacobian matrix of Eqn. (5) at a fixed point $\overline{X} = (\bar{x}_1, \ldots, \bar{x}_n)$ for internal, boundary or vertices points,

$$J_X = \mathrm{diag}(k_1, \ldots, k_n) + \frac{1}{\epsilon}\mathrm{diag}(\bar{x}_1(1-\bar{x}_1), \ldots, \bar{x}_n(1-\bar{x}_n))W \qquad (7)$$

where

$$k_i = \left\{ \begin{array}{cc} \infty & \text{if } k < 1, \bar{x}_i = 0 \text{ or } 1 \\ e^{\frac{1-2\bar{x}_i}{\epsilon}[\sum_{j=1}^{n} \omega_{ij}\bar{x}_j + a_i - \omega_{ii}a_{0i}]} & \text{if} k = 1, \bar{x}_i = 0 \text{ or } 1 \\ 0 & \text{if } k > 1, \bar{x}_i = 0 \text{ or } 1 \\ k & \text{if } \bar{x}_i \neq 0 \text{ or } 1 \end{array} \right\}. \qquad (8)$$

Then by checking the eigenvalues of J_X, we have the following results.

Theorem 4. Assume $W = W^T$.

1. For synchronous updating, a fixed point $\overline{X}$ in the region S is asymptotically stable if the largest eigenvalue $\lambda'_{\max}$ and the smallest eigenvalue $\lambda'_{\min}$ for the matrix J_X of Eqn. (7) satisfy $\lambda'_{\max} < 1$ and $\lambda'_{\min} > -1$; on the other hand, a fixed point $\overline{X}$ in the region S is unstable if either $\lambda'_{\max} > 1$ or $\lambda'_{\min} < -1$ is satisfied.
2. For asynchronous updating, a fixed point $\overline{X}$ in the region S is asymptotically stable if $1 > \max_{i=1}^{n}[k_i + \frac{\omega_{ii}}{\epsilon}\bar{x}_i(1-\bar{x}_i)]$ and $\min_{i=1}^{n}[k_i + \frac{\omega_{ii}}{\epsilon}\bar{x}_i(1-\bar{x}_i)] > -1$.

On the other hand, for a special asynchronous updating scheme, i.e., cyclic updating, we can derive a stricter theorem. Let U_a, D_a and L_a be an upper triangular, a diagonal and a lower triangular parts of J_X defined in Eqn. (7), i.e., $J_X = U_a + D_a + L_a$. Define λ_{cmin} and λ_{cmax} to be the minimal and maximal eigenvalues of the matrix $(E - L_a)^{-1}(U_a + D_a)$, respectively.

Theorem 5. Assume $W = W^T$. For cyclic updating,

1. a fixed point $\overline{X}$ in the region S is asymptotically stable if the largest eigenvalue λ_{cmax} and the smallest eigenvalue λ_{cmin} of the matrix $(E - L_a)^{-1}(U_a + D_a)$ satisfy $|\lambda_{\mathrm{cmax}}| < 1$ and $|\lambda_{\mathrm{cmin}}| < 1$;
2. on the other hand, a fixed point $\overline{X}$ in the region S is unstable if either $|\lambda_{\mathrm{cmax}}| > 1$ or $|\lambda_{\mathrm{cmin}}| > 1$ is satisfied;
3. when $k < 1$, no point on vertices or boundary points is asymptotically stable.

4.2. Relations of Stable Fixed Points and Local Minima

Since $f(X)$ is usually used as an objective function when applying to optimization problem, we are interested in the relations between the stable fixed points of Eqn. (5) and minima of $f(X)$. We first classify the stationary points of $f(X)$, the local minima of $f(X)$ and the fixed points of Eqn. (5), and then derive their relations in this subsection.

Definition 8. A state X_s is said to be a stationary point of $f(X)$ if satisfying $\frac{\partial f(X_s)}{\partial X} = 0$.

Therefore, stationary points of $f(X)$ are composed of the following three sets

- local minima where f_{XX} is semi-positive definite
- local maxima where f_{XX} is semi-negative definite
- saddle points where f_{XX} is indefinite

It should be noticed that some of local minima for $f(X)$ subject to $E_0 \leq X \leq E_1$ may not belong to the stationary points of $f(X)$. Namely, local minima of $f(X)$ subject to $E_0 \leq X \leq E_1$ include

- the local minima on the internal points of X, which belong to the stationary points of $f(X)$,
- the local minima on the boundary points or vertices of X, which may not belong to stationary points of $f(X)$.

Furthermore regardless of synchronous or asynchronous updating, fixed points of Eqn. (5) are composed of

- all stationary points of $f(X)$ (i.e., local minima, local maxima and saddle points of $f(X)$),
- all of the vertices of X.

For the cases satisfying Theorem 2 or 3, where only the fixed points belonging to the local minima of $f(X)$ are stable, we have the following concise results.

Theorem 6. Assume the same conditions as Theorem 2 or Theorem 3. Then a point X is asymptotically stable for Eqn. (5) if and only if this point is a local minimum of $f(X)$ for $E_0 \leq X \leq E_1$.

In particular, when $\omega_{ii} = 0$ for $i = 1, \ldots, n$ and $k = 1$, all local minima of $f(X)$ are on the vertices. Therefore, assuming the same conditions of Theorem 2 or Theorem 3 and $\omega_{ii} = 0$ for $i = 1, \ldots, n$ and $k = 1$, a point X is asymptotically stable for Eqn. (5) if and only if this point is a local minimum of $f(X)$ for $E_0 \leq X \leq E_1$ on the vertices.

According to Eqn. (7), the closer $\bar{x}_i (i = 1, \ldots, n)$ are to 0 and 1, the smaller the absolute values of eigenvalues $\lambda'_{\min}$ and $\lambda'_{\max}$ are. Therefore, this implies that the system is likely to asymptotically converge to local minima near the vertices for $-1 < k < 1$.

For more general cases, we can derive the following conclusions by Theorems 4-5.

Theorem 7. Assume $W = W^T$.

1. In the internal points of X, only local minima of $f(X)$ can be asymptotically stable for Eqn. (5), for both synchronous and asynchronous updating.
2. As far as $k \neq 1$, there is no stationary points of $f(X)$ on the vertices or boundary points of X.
3. When $k < 1$, no point on vertices or boundary points is asymptotically stable for both asynchronous and synchronous updating.
4. When $k > 1$, all vertices are asymptotically stable for both synchronous and asynchronous updating.

5. For both synchronous updating and cyclic updating, when $k = 1$, a vertex $\overline{X}$ is asymptotically stable if $\max_{i=1}^{n} e^{\frac{1-2\bar{x}_i}{\epsilon}[\sum_{j=1}^{n} \omega_{ij}\bar{x}_j + a_i - \omega_{ii}a_{0i}]} < 1$; on the other hand, a vertex is unstable if $\max_{i=1}^{n} e^{\frac{1-2\bar{x}_i}{\epsilon}[\sum_{j=1}^{n} \omega_{ij}\bar{x}_j + a_i - \omega_{ii}a_{0i}]} > 1$.

6. For random-order updating, when $k = 1$, a vertex is asymptotically stable if $\max_{i=1}^{n} e^{\frac{1-2\bar{x}_i}{\epsilon}[\sum_{j=1}^{n} \omega_{ij}\bar{x}_j + a_i - \omega_{ii}a_{0i}]} < 1$.

Theorems 6-7 are useful for both designs of neural networks and applications of CNN and TCNN to practical problems, e.g., optimization or associative memory, because they indicate which local minimum of the computational energy function is an attractor and which one is not.

4.3. Local Bifurcations

Next, let us examine the bifurcations of the fixed points, in particular of the stationary points of $f(X)$, by taking P as bifurcation parameters. P is a parametric column vector defined by $P = (\omega_{11}, \ldots, \omega_{nn})^T$ or other parameters in Eqn. (4).

Since many applications like combinatorial optimization and associative memory take minima of $f(X)$ as targets of information processing, we focus on analysing bifurcations for the stationary points of $f(X)$, especially for internal fixed points. Instead of Eqn. (5), we use Eqn. (4) in the following analysis of the codimension-one bifurcations of internal points, by applying the center manifold method [12, 14, 17].

Eqn. (4) is rewritten as,

$$Y(t+1) = Y(t) - \frac{\partial f(X(Y(t));P)}{\partial X} = Y(t) - f_X(X(Y(t));P), \qquad (9)$$

where $f(X;P)$ is defined in Eqn. (6) with parameters P expressed explicitly, and $f_X = \frac{\partial f}{\partial X}$.

Then the Jacobian matrix $J_Y(X(Y(t));P) = \frac{\partial Y(t+1)}{\partial Y(t)}$ of Eqn. (9) at $(X(t), P)$ becomes

$$J_Y = E - f_{XY}(X(t);P) = E - \frac{1}{\epsilon}\mathrm{diag}\Big[x_1(t)(1-x_1(t)), \ldots, x_n(t)(1-x_n(t))\Big] f_{XX}(X(t);P) \qquad (10)$$

where $f_{XY} = \frac{\partial f_X}{\partial Y}, f_{XX} = \frac{\partial f_X}{\partial X}$ and f_{XX} is a symmetrical matrix. Due to this symmetrical structure of f_{XX} or J_Y, we immediately have the following remark.

Remark 4.1. Assume $W = W^T$, and take P as bifurcation parameters. Then there is no Neimark-Sacker bifurcation for fixed points of Eqn. (4). In other words, there only exist two types of generic codimension-one bifurcations, i.e., fold bifurcations (saddle-node bifurcations) and flip bifurcations (period-doubling bifurcations). Moreover all fixed point bifurcations occur at the stationary points of $f(X)$.

Let $\overline{Y}$ or $\overline{X}$ ($\bar{x}_i = \frac{1}{1+e^{-\bar{y}_i/\epsilon}}$ for $i = 1, \ldots, n$) be an internal fixed point, which is in fact a stationary point of $f(X; P)$ for $W^T = W$. Let P^c be the critical values for bifurcations. Then $\frac{\partial f(X;P)}{\partial X}$ can be expanded in the vicinity of $(\overline{Y}, P^c)$ in Taylor series with respect to both variables Y and parameters P, such that Eqn. (9) becomes,

$$\begin{aligned} Y(t+1) \;=\; & Y(t) - \Big[f_{XY}\Delta Y + \frac{1}{2} f_{XYY}(\Delta Y \cdot \Delta Y) \\ & + \frac{1}{6} f_{XYYY}(\Delta Y \cdot \Delta Y \cdot \Delta Y) + \cdots \Big] \\ & - [f_{XP}\Delta P + f_{XPY}(\Delta P \cdot \Delta Y) + \cdots] - \ldots, \end{aligned} \tag{11}$$

where, $\Delta Y = Y(t) - \overline{Y}$, $\Delta P = P - P^c$ and $f_{XYY}(\overline{X}; P^c)$ is a block matrix [32].

Since $\overline{X}$ or $\overline{Y}$ is a fixed point and P^c are the critical values of bifurcations, $f_X(\overline{X}; P^c) = 0$ and at least one of eigenvalues for the Jacobian $J_Y(\overline{X}; P^c)$ at $(\overline{X}, P^c)$ is $\lambda = 1$ or $\lambda = -1$.

Therefore, by applying the center manifold method to $\lambda = 1$ or $\lambda = -1$, the following theorems hold according to bifurcation theory [8, 12].

Theorem 8. Let $W^T = W$. Assume that $\overline{X}$ is an internal fixed point of Eqn. (9) or Eqn. (4) and that $f_{XY}(\overline{X}; P^c)$ has one zero eigenvalue with the right eigenvector U and the left eigenvector $U^\dagger$. Then for synchronous updating, a fold bifurcation (saddle-node bifurcation) occurs if

$$U^\dagger f_{XYY}(\overline{X}; P^c)(U \cdot U) \neq 0$$

and

$$U^\dagger f_{XP}(\overline{X}; P^c) \neq 0.$$

Since the bifurcations in Theorem 8 occur when $f_{XX}(\overline{X}; P^c)$ has a zero eigenvalue and $f_X(\overline{X}; P^c) = 0$, they are actually stationary point bifurcations of $f(X; P)$.

Remark 4.2. The fold bifurcations for fixed points in Theorem 8 generate or eliminate a pair of stationary points of $f(X; P)$.

Theorem 9. Let $W^T = W$. Assume that $\overline{X}$ is an internal fixed point of Eqn. (9) or Eqn. (4) and that $\mathrm{diag}(\bar{x}_1(1-\bar{x}_1), \ldots, \bar{x}_n(1-\bar{x}_n)) f_{XX}(\overline{X}; P^c)$ has one eigenvalue 2ϵ with the right eigenvector V and the left eigenvector $V^\dagger$.

Then for synchronous updating, a flip bifurcation (a period-doubling bifurcation) occurs if

$$\left[\frac{V^\dagger}{2} f_{XYY}(\overline{X}; P^c)(V \cdot V)\right]^2 - \frac{V^\dagger}{6} f_{XYYY}(\overline{X}; P^c)(V \cdot V \cdot V) \neq 0$$

and

$$2V^\dagger f_{XPY}(\overline{X}; P^c)V - [V^\dagger f_{XP}(\overline{X}; P^c)][V^\dagger f_{XYY}(\overline{X}; P^c)(V \cdot V)] \neq 0.$$

Since the flip bifurcations in Theorem 9 occur when $f_{XX}(\overline{X}; P^c)$ has a positive eigenvalue and $f_X(\overline{X}; P^c) = 0$, these bifurcations only stabilize or unstabilize the local manifold along the direction of the eigenvector corresponding to the positive eigenvalue of $f_{XX}(\overline{X}; P^c)$. Therefore, the flip bifurcations do not generally create or eliminate any stationary point of $f(\overline{X}; P)$, but may unstabilize or stabilize local minima of $f(X; P)$.

Remark 4.3. The flip bifurcations for fixed points in Theorem 9 occur at either a saddle point or a local minimum of $f(X; P)$. The flip bifurcations for fixed points cannot create (or eliminate) stationary points of $f(\overline{X}; P)$ although it may cause a change of stability of a fixed point together with the birth of a new period-two orbit.

Since one eigenvalue of f_{XX} is positive for the flip bifurcation, the flip bifurcation only occurs at a local minimum or a saddle point of $f(X; P)$. As the fold and flip bifurcations are generic, most bifurcations can be expected to be fold bifurcations or flip bifurcations when $\|P\|$ is reduced from a sufficiently large value to zero if we define $P = (\omega_{11}, \ldots, \omega_{nn})^T$ as temperature parameters to realize annealing of Eqn. (4) or Eqn. (5) [7]. However, if a dynamical system has certain symmetries, the other bifurcations can also be observed, such as pitchfork bifurcations and others.

In this subsection, we have examined only the bifurcations of synchronous updating. However, the bifurcation conditions of asynchronous updating can be derived analogously to those of synchronous updating.

Clearly, all of analyses in this section are independent of the values of I and $(a_{01}, \ldots, a_{0n})$ in Eqn. (5). If $I_0 = E_0$, Eqn. (4) or Eqn. (5) is equivalent to chaotic neural networks (CNNs)[4]. Therefore, all of theoretical results in this section also hold for CNNs.

5. Global Structure and Topological Chaos

In contrast to local analysis in Section 4, we examine global structure of TCNNs in this section. For the sake of the simplification of description, we assume that all the self-recurrent connection weights take the same value, i.e., $\omega_{ii} = \omega$ for $i = 1, \ldots, n$, and dynamics for $Y(t)$ or $X(t)$ is based on synchronous updating, in the subsequent sections. All of the proofs of the theorems in this section are provided in [8, 10].

In this section, we first examine all possible fixed points in TCNNs, and then focus on a unique repeller, which is actually a snap-back repeller, thereby proving the existence of topological chaos.

5.1. Existence of a Snap-back Repeller and Topological Chaos

To derive all possible fixed points, define

$$\begin{aligned} d_{i,\max} &= \sum_{j=1, j\neq i}^{n} \omega_{ij} u(\omega_{ij}) + a_i \\ d_{i,\min} &= \sum_{j=1, j\neq i}^{n} \omega_{ij} u(-\omega_{ij}) + a_i \end{aligned}$$

where $u(\omega_{ij}) = 1$ for $\omega_{ij} > 0$ and 0 for $\omega_{ij} \leq 0$. Let

$$d_i(X) = \sum_{j=1, j\neq i}^{n} \omega_{ij} x_j(t) + a_i$$

which is independent of $x_i(t)$ and bounded because $d_{i,\min} < d_i(X) < d_{i,\max}$ due to $0 < x_j(t) < 1$. Then Eqn. (2) or (4) can be rewritten as

$$y_i(t+1) = f_i(Y(t)) = ky_i(t) + d_i(X) + \omega(x_i(t) - a_{0i})i = 1, \dots, n. \quad (12)$$

At a fixed point, Eqn. (12) can be changed as $\frac{(1-k)y_i - d_i(X)}{\omega} = x_i - a_{0i}$ due to $y_i(t+1) = y_i(t) = y_i$. If y_i takes a finite value (i.e., y_i are bounded) as $|\omega| \to \infty$, then it is evident that $x_i \simeq a_{0i}$, i.e., $y_i \simeq \epsilon ln \frac{a_{0i}}{1-a_{0i}}$. On the other hand, we can rewrite Eqn. (12) as $y_i = \frac{\omega(x_i - a_{0i}) + d_i(X)}{1-k}$ at a fixed point. If y_i takes an infinite value at a fixed point as $|\omega| \to \infty$, there are two possible values for x_i, i.e., $x_i = 1$ corresponding to $y_i \simeq \frac{\omega(1-a_{0i}) + d_i(X)}{1-k} \to +\infty$, and $x_i = 0$ corresponding to $y_i \simeq \frac{-\omega a_{0i} + d_i(X)}{1-k} \to -\infty$. Therefore, considering $x_j = a_{0j}$ or 1 or 0, let $d_{i,ext}(X) = \sum_{j=1, j\neq i}^{n} \omega_{ij} v(x_j) + a_i$ where

$$v(x_j) = \begin{cases} 1 & , x_j > a_{0j} \\ a_{0j} & , x_j = a_{0j} \\ 0 & , x_j < a_{0j}. \end{cases}$$

Define $y_i^1 = \epsilon ln \frac{a_{0i}}{1-a_{0i}}$, $y_i^2 = \frac{\omega(1-a_{0i}) + d_{i,ext}(X)}{1-k}$, $y_i^3 = \frac{-\omega a_{0i} + d_{i,ext}(X)}{1-k}$ for $i = 1, \dots, n$, and let $Y^1 = (y_1^1, \dots, y_n^1)^T$. Then there exist many fixed points in the neighbourhoods of the points Y with each elements y_i^1, y_i^2 or y_i^3.

Theorem 10. Assume $1 > a_{0i} > 0$ for $i = 1, \dots, n$, and $|\omega|$ is sufficiently large.

1. If $k > 1, \omega < 0$ or $k < -1, \omega > 0$, then Eqn. (4) has 3^n fixed points, which are all repellers, in a neighbourhood of $Y = (y_1, \dots, y_n)^T$ where y_i is equal to y_i^1 or y_i^2 or y_i^3 for $i = 1, \dots, n$;
2. if $-1 \leq k < 1, \omega > 0$, then Eqn. (4) has 2^n attracting fixed points in a neighbourhood of $Y = (y_1, \dots, y_n)^T$ where y_i is equal to y_i^2 or y_i^3 for $i = 1, \dots, n$, and a repeller in a neighbourhood of $Y^1 = (y_1^1, \dots, y_n^1)^T$, and $3^n - 2^n - 1$ saddle fixed points;
3. if the conditions 1 and 2 are not satisfied (i.e., $k \leq 1, \omega < 0$ or $k \geq 1, \omega > 0$), then Eqn. (4) has one and only one fixed point, which is a repeller, in a neighbourhood of $Y^1 = (y_1^1, \dots, y_n^1)^T$.

When one of three conditions of Theorem 10 is satisfied, there exists a bounded fixed point $\overline{Y}^1 = (\bar{y}_1^1, \dots, \bar{y}_n^1)^T$ in a neighbourhood of Y^1, where $\bar{y}_i^1$ is defined as

$$\bar{y}_i^1 = \epsilon ln \frac{a_{0i}}{1 - a_{0i}} + \mathcal{O}(\frac{1}{\omega}) \quad \text{for } i = 1, \dots, n \tag{13}$$

In other words, there exists at most 3^n fixed points in TCNNs but only one bounded fixed point $\overline{Y}^1$ when $|\omega|$ is sufficiently large.

By examining the unique fixed point $\overline{Y}^1$, we have the following theorem on the existence of topological chaos for TCNNs in the sense of Marotto [28].

Theorem 11. Assume $1 > a_{0i} > 0$ for $i = 1, \dots, n$ and that $|\omega|$ is sufficiently large. Then Eqn. (4) has a fixed point $\overline{Y} = (\bar{y}_1, \dots, \bar{y}_n)$ which can be expressed as $\bar{y}_i = \epsilon \ln \frac{a_{0i}}{1-a_{0i}} + \mathcal{O}(\frac{1}{\omega})$ for $i = 1, \dots, n$. If the parameters a_{0i}, k and ω further satisfy one of the following three conditions for each i $(i = 1, \dots, n)$,

1. $\omega < 0, k > \frac{a_{0i}}{1-a_{0i}}$,
2. $\omega < 0, k > \frac{1-a_{0i}}{a_{0i}}$,
3. $k < -1, \omega > 0$,

then $\overline{Y}$ is a snap-back repeller for Eqn. (4). In other words, Eqn. (4) is chaotic in the sense of Marotto.

Theorem 11 implies that TCNNs have topologically chaotic structure including a scrambled set and infinitely many periodic points as indicated in (A), (B) and (C) of Theorem 1.

In particular, we can show that there exists at least one snap-back point Y^0 in a neighbourhood of $(y_1, \dots, y_n)$ where $y_i = \ln \frac{a_{0i}(1+1/k)}{1-a_{0i}(1+1/k)}$ or $\ln \frac{a_{0i}(1+1/k)-1/k}{1-a_{0i}(1+1/k)+1/k}$, such that $F^2(Y^0) = \overline{Y}$ if the conditions of Theorem 11 are satisfied. In other words, according to all combination of y_i there are possibly 2^n homoclinic orbits in TCNNs.

5.2. Attracting Set with Homoclinic Orbits

Theorem 11 ensures that TCNNs have complicated structure including a scrambled set and a snap-back repeller. The following theorem[10] further gives sufficient conditions under which a homoclinic orbit or a snap-repeller is at least contained in an attracting set.

Theorem 12. Assume $1 > a_{0i} > 0$ for $i = 1, \ldots, n$ and $|\omega|$ is sufficiently large. Assume that ω, k and a_{0i} satisfy one of the following three conditions for each i $(i = 1, \ldots, n)$,

1. $\omega < 0$ and $\min\{\frac{1}{1-a_{0i}}, \frac{1}{a_{0i}}\} > k > \min\{\frac{a_{0i}}{1-a_{0i}}, \frac{1-a_{0i}}{a_{0i}}\}$,
2. $\omega < 0$ and $1 \geq k > \min\{\frac{a_{0i}}{1-a_{0i}}, \frac{1-a_{0i}}{a_{0i}}\}$,
3. $\omega > 0$ and $-1 > k > \max\{\frac{-1}{1-a_{0i}}, \frac{-1}{a_{0i}}\}$.

Then there is an attracting set Λ satisfying $W^u(\overline{Y}) \subset \Lambda$ for Eqn. (4) and containing the transversal homoclinic orbits of $\overline{Y}$, where

- $\Lambda \subset (\omega(1 - a_{01}) + d_{1,\min}, -\omega a_{01} + d_{1,\max}) \times \cdots \times (\omega(1 - a_{0n}) + d_{n,\min}, -\omega a_{0n} + d_{n,\max})$ for condition 1 or 2;
- $\Lambda \subset (-\omega a_{01} + d_{1,\min}, \omega(1 - a_{01}) + d_{1,\max}) \times \cdots \times (-\omega a_{0n} + d_{n,\min}, \omega(1 - a_{0n}) + d_{n,\max})$ for condition 3.

In other words, Marotto chaos is included in an attracting set under the conditions of Theorem 12.

6. Strange Attractor and Globally Searching Ability

The Marotto chaos is formal chaos which may not be numerically observed when the scrambled set or the homoclinic orbit is not appropriately included in an attracting set. On the other hand, although the points $F^m(Y)$ representing the system behaviour should eventually converge to an attracting set, there may exist attractors, which are smaller than the attracting set, i.e., an attracting set may contain one or several attractors as proper subsets. In this section, extending the results of Theorem 12, we will first show that the homoclinic orbits of $\overline{Y}$ are actually included in an attractor which is indecomposable, by proving the topological transitivity for TCNNs; this means the existence of the strange attractor. Then we explain the global searching ability of TCNNs[10, 11] by examining the region covered by the strange attractor.

6.1. Strange Attractor

We only give a brief outline of the proof for the strange attractor in TCNNs (see [11] for the details). The main procedures to prove the existence of the strange attractor are as follows:

- for the one-dimensional case, we first show that the interval $\Lambda = [y_{\min}, y_{\max}]$ is topologically transitive by using the Schwarzian derivative when $|\omega|$ is sufficiently large, and then prove that $\Lambda = W^u(\bar{y}^1)$ is a strange attractor;
- for the $n-$dimensional TCNNs, we prove that there exists a integer $m > 0$ such that $\overline{Y} \in F^m(U)$ for any open set $U \subset W^u(\overline{Y})$ by constructing two one-dimensional associated TCNNs, and then show that $W^u(\overline{Y}^1)$ is topologically transitive when $|\omega|$ is sufficiently large;
- we finally prove that F is sensitively dependent on initial conditions on $W^u(\overline{Y})$.

$$\begin{aligned} y_{\min} &= k\epsilon \ln \frac{x^{\text{emin}}}{1-x^{\text{emin}}} + \omega(x^{\text{emin}} - a_0) + d = k\epsilon ln \frac{-\omega}{k\epsilon} \\ &\quad +\omega(1-a_0) + k\epsilon + d + \mathcal{O}\left(\frac{1}{\omega}\right) \qquad (14) \\ y_{\max} &= k\epsilon \ln \frac{x^{\text{emax}}}{1-x^{\text{emax}}} + \omega(x^{\text{emax}} - a_0) + d = -k\epsilon ln \frac{-\omega}{k\epsilon} \\ &\quad -\omega a_0 - k\epsilon + d + \mathcal{O}(\frac{1}{\omega}). \qquad (15) \end{aligned}$$

where $x^{\text{emin}} = \frac{1+\sqrt{1+4k\epsilon/\omega}}{2}$ and $x^{\text{emax}} = \frac{1-\sqrt{1+4k\epsilon/\omega}}{2}$.

In general, the topologically transitivity of a set does not imply that its orbits are also sensitively dependent on initial conditions[15]. Therefore, we need to prove sensitive dependence on initial conditions.

Theorem 13. Assume $k > 1$ and condition 1 of Theorem 12. Then there exists a strange attractor in n-dimensional TCNNs, which is equal to $W^u(\overline{Y})$ and $\Lambda = W^u(\overline{Y})$, provided that $|\omega|$ is sufficiently large where Λ is the attracting set of Theorem 12.

The strange attractor in this article has the following properties.

Remark 6.1. Assume that Γ is the strange attractor of F satisfying the conditions of Theorem 13. Then

1. Γ is asymptotically stable,
2. F is topologically transitive on Γ,
3. F is chaotic in the sense of Marotto,
4. F is sensitively dependent on initial conditions on Γ.

1 and 2 of Remark 6.1 can be easily derived from the definitions of asymptotical stability and a strange attractor. 3 of Remark 6.1 is from Theorem 11.4 of Remark 6.1 describes an unpredictable property of a strange attractor. Therefore, for n-dimensional TCNNs, the global unstable set $W^u(\overline{Y})$ is a strange attractor which is asymptotically stable, sensitively dependent on initial conditions and topologically transitive according to Remark 6.1. In addition, due to the snap-back repeller $\overline{Y}^1$, this strange attractor has the structure of Marotto chaos according to Theorem 11.

6.2. Global Searching Ability

Since the strange attractor in TCNNs is actually $W^u(\overline{Y})$, which contains all of the bounded region when $|\omega| \to \infty$, there are 2^n sufficiently large snap-back points for the n-dimensional system, whose number is equal to the combination number of n-binary variables for the combinatorial optimization problem[10].

In applications of neural computation, a Lyapunov function, or a computational energy function e.g. $f(X)$, plays an important role because minima of such an energy function correspond to possible solutions of combinatorial optimization problems, stored patterns of associative memory and so on, as demonstrated in [20, 21, 22, 23, 24, 26, 33, 36, 37, 38, 39].

Actually the attractor Λ for $Y(t)$ is $\Lambda \simeq [\omega(1-a_{01}), -\omega a_{01}] \times \cdots \times [\omega(1-a_{0n}), -\omega a_{0n}]$ for $\omega < 0$ or $\simeq [-\omega a_{01}, \omega(1-a_{01})] \times \cdots \times [-\omega a_{0n}, \omega(1-a_{0n})]$ for $\omega > 0$. As $|\omega| \to \infty$, the attractor covers all the bounded regions for $Y(t)$. Moreover, according to eqns.(14)–(15), the attractor Λ_X for $X(t)$ is $\Lambda_X = V_1^x \times \cdots \times V_n^x$ where for $i = 1, \ldots, n$

$$V_i^x = \left[\frac{1}{e^{-\frac{\omega(1-a_{0i})}{\epsilon}}} + \mathcal{O}\left(\frac{1}{e^{-\frac{2\omega(1-a_{0i})}{\epsilon}}}\right), 1 - \frac{1}{e^{-\frac{\omega a_{0i}}{\epsilon}}} + \mathcal{O}\left(\frac{1}{e^{-\frac{2\omega a_{0i}}{\epsilon}}}\right)\right] \quad (16)$$

for $\omega < 0$, or

$$V_i^x = \left[\frac{1}{e^{\frac{\omega a_{0i}}{\epsilon}}} + \mathcal{O}\left(\frac{1}{e^{\frac{2\omega a_{0i}}{\epsilon}}}\right), 1 - \frac{1}{e^{\frac{\omega(1-a_{0i})}{\epsilon}}} + \mathcal{O}\left(\frac{1}{e^{\frac{2\omega(1-a_{0i})}{\epsilon}}}\right)\right] \tag{17}$$

for $\omega > 0$, where Λ_X is transformed from Λ by Eqn. (3).

That is, the attractor Λ_X for $X(t)$ of Eqn. (4) is arbitrarily close to $[0,1]^n$ when $|\omega|$ is sufficiently large. On the other hand, according to Lemma 3.1 of [8], there exist $b_{\min}$ and $b_{\max}$ such that all the minima of $f(X)$ are in the open region $(b_{\min}, b_{\max})^n$ where $0 < b_{\min} < b_{\max} < 1$ independent of ω when $k \neq 1$, which means that all minima including global minima of $f(X)$ are in the attractor Λ_X of Eqn. (4) [10, 11].

Theorem 14. Assume the same conditions of Theorem 12, then all the minima including global minima of $f(X)$ of Eqn. (6) are in the attractor Λ_X of Eqn. (4) when $k \neq 1$.

Even if $k = 1$ for which all the minima are on the vertices $\{(x_1, \ldots, x_n) \, | x_i = 0 \text{ or } 1; i = 1, \ldots, n\}$ of S according to Corollary 3.1 of [8], the attractor Λ_X is arbitrarily close to $[0,1]^n$ from Eqns. (16)–(17) when $|\omega|$ is sufficiently large. Therefore, TCNN has ability for global search when $|\omega|$ is sufficiently large.

Furthermore, if $\overline{Y}^1$ is in an attractor and both of the conditions 1 and 2 or the condition 3 of Theorem 12 are satisfied, $Y(t)$ in the homoclinic orbit should pass the 2^n homoclinic points $Y^s = (y_1, \ldots, y_n)^T$ where $y_i = \frac{\omega a_{0i}}{k} + \mathcal{O}(1)$ or $\frac{-\omega(1-a_{0i})}{k} + \mathcal{O}(1)$ which implies that $X(t)$ in the homoclinic orbit passes the neighbourhoods of all vertices $\{(x_1, \ldots, x_n) | x_i = 0 \text{ or } 1; i = 1, \ldots, n\}$ of S which is also the state space of the general combinatorial problems, when $|\omega|$ is large enough.

In contrary to locally convergent dynamics reined by a Lyapunov stability [10], globally chaotic dynamics of TCNNs shown in this article is not restricted to any local area, but able to visit every possible state in the strange attractor, i.e., TCNNs have the global searching ability. For instance, a TCNN can be used as an optimizer to solve a combinatorial optimization problem because the strange attractor of eqns.(2)-(3) encloses all global minima of $f(X)$ when $|\omega|$ is sufficiently large. Furthermore, such global dynamics of the strange attractor also provides a basis for dynamical associative memory which can search all of stored patterns or generate rich spatio-temporal patterns composed of spatial patterns stored with synaptic connection weights[1, 2].

For the global analysis in sections 5 and 6, we only examine the dynamics of TCNNs for sufficiently large $|\omega|$, although chaotic dynamics is observed when $|\omega|$ takes a moderate value, for which TCNNs may have quite different topological structures. In other words, it is still unclear for global structures for TCNNs with a moderate $|\omega|$ and needs to be further analysed intensively.

7. Chaotic Simulated Annealing

As shown in the previous section, TCNNs have the global searching ability without getting stuck at local minima of a computational energy function. However, this ability can cause problems because TCNNs escape even from global minima. To copy with the defect, a kind of deterministic annealing was proposed where some parameters, for example, the self-recurrent connection weight is gradually reduced with time as follows [7]:

$$\omega(t+1) = (1-\beta)\omega(t) \tag{18}$$

where $1 > \beta > 0$ and $\omega_{ii} = \omega$ for $i = 1, \ldots, n$.

The annealing procedure of TCNNs is fully deterministic, namely, starting from deterministically chaotic dynamics with sufficiently large $|\omega|$, through a reversed period-doubling route with decreasing the value of $|\omega|$ which corresponds to the temperature of usual annealing, finally reaching a stable equilibrium solution. We call this mechanics of TCNNs as *chaotic simulated annealing (CSA)* which is regarded as a kind of *deterministic simulated annealing (DSA)* [7]. Compared with the stochastic thermal fluctuations of SSA, chaos can be viewed as deterministic noise and the convergence process of CSA is characterized as the nonlinear bifurcations which drive the neurodynamics to converge from strange attractors to an attracting fixed point. Moreover, modified CSA with adaptive annealing is also proposed [33].

Many experiment works have shown that CSA has very high ability to reach global minima with striking efficiency for various optimization problems [7, 23, 24, 26, 33, 36, 37, 38, 39].

8. Applications to Optimization

Traveling salesman problem (TSP) is a NP-hard combinatorial optimization problem, which seeks the shortest route through n cities, visiting each once and only once and returning to the starting city [25]. TSP can be generally divided into symmetric TSP with $d_{ij} = d_{ji}$, and asymmetric TSP with $d_{ij} \neq d_{ji}$, where d_{ij} is the distance from city-i to city-j. For symmetric TSP, there are $(n-1)!/2$ possible tours (or feasible solutions), while there are $(n-1)!$ possible tours for asymmetric TSP. Clearly, the number of possible tours grows faster than any finite power of n. As opposed to intuition, asymmetric TSP is actually easier to solve than symmetric TSP in cases such that distances are randomly generated. TSP is simple to state but difficult to solve exactly due to a large number of local minima.

Since Hopfield and Tank applied their neural networks to TSP [22], TSP has been intensively studied in the field of artificial neurocomputing [7]. In this article, Symmetric TSP on the plane is used to test the ability of TCNN with CSA as a global optimization method for combinatorial problems. The detail description and simulation for this section are shown in [7].

We adopt the formulation for TSP by Hopfield and Tank [22]. Namely, a solution of TSP with n cities is represented by a $n \times n$ - permutation matrix, where each element corresponds to output of a neuron in a network with $n \times n$ lattice structure. Assume x_{ij} to be the neuron output which represents to visit city-i in visiting order-j. A computational energy function which is to minimize the total tour length while simultaneously satisfying all constraints takes the following form:

$$\begin{aligned} f(X) &= \frac{W_1}{2}\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij} - 1\right)^2 + \sum_{j=1}^{n}\left(\sum_{i=1}^{n} x_{ij} - 1\right)^2\right\} \\ &\quad + \frac{W_2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_{kj+1} + x_{kj-1})x_{ij}d_{ik} \qquad (19) \end{aligned}$$

where $x_{i0} = x_{in}$ and $x_{in+1} = x_{i1}$. W_1 and W_2 are the weight parameters corresponding to the constraints and the cost function of the tour length, respectively, where d_{ij} is the distance between city-i and city-j.

By setting each connection weight ω_{ij} as the same as the Hopfield-Tank formulation, the difference equations describing the network dynamics of TCNN for the TSP are obtained from eqns.(2)-(4) as follows:

$$x_{ij}(t) = \frac{1}{1 + e^{-y_{ij}(t)/\epsilon}} \qquad (20)$$

$$y_{ij}(t+1) = ky_{ij}(t) + \omega(t)(x_{ij}(t) - a_0) - W_1\left(\sum_{l\neq j}^{n} x_{il}(t) + \sum_{k\neq i}^{n} x_{kj}(t)\right)$$

$$-W_2\left(\sum_{k\neq i}^{n} d_{ik}x_{kj+1}(t) + \sum_{k\neq i}^{n} d_{ik}x_{kj-1}(t)\right) + W_1 \qquad (21)$$

$$\omega(t+1) \;=\; (1-\beta)\omega(t)(i,j=1,\ldots,n) \qquad (22)$$

where the variable $\omega(t)$ is assumed to be common to all the neurons, and $a_{ij} = W_1$.

In the numerical simulations, $k = 0.9, W_1 = W_2 = 0.015, \omega(0) = 0.08$, $\epsilon = 1/250$ and $a_{0ij} = a_0 = 0.65$ for $i, j = 1, \ldots, n$, and states of neurons are cyclically updated.

We have analysed 10 city TSP of the data [22] with $\beta = 0.015, 0.010, 0.005$, 0.003, and with 5000 different initial conditions of y_{ij} generated randomly in the region [-1,1] for each value of β [7].

As a result [7], TCNN converges averagely with 81 iterations when $\beta = 0.015$, where among 5000 cases, 54 cases converge either to local minima (tour length = 2.74,2.78,...) or to infeasible solutions, and 4946 cases are to global minima (Tour length=2.70). That is, the rate of global optimization is 98.9% for $\beta = 0.015$. The different initial conditions of y_{ij} can derive different output representations of global optimal solutions with the same tour length.

The parameter β can be considered as a damping speed parameter of the negative self-feedback strength or the "temperature" $|\omega|$, which controls the annealing schedule. In order to produce stronger and longer chaotic dynamics, we set β small as $\beta = 0.005$ and 0.003, so that the temperature $|\omega|$ decreases more slowly. Then, the results are considerably improved, e.g., when $\beta = 0.005$, TCNN converges averagely with 234 iterations and all the obtained solutions are feasible. Besides, the rate of global optimization reaches 99.9%, which is extremely high compared with the rate less than 37% obtained by the Hopfield-Tank neural network [7]. Furthermore, 100% of the rate of global optimization is also realized by simulations with $\beta = 0.003$ for this toy problem with 10 cities.

On the other hand, a kind of SSA with the Metropolis algorithm is also applied to the same 10 city TSP to compare with CSA in terms of computational cost and accuracy. In this calculation, SSA adopts the same cooling schedule as that of CSA, namely the exponential cooling schedule of Eqn. (22) with $\beta = 0.00004$ for annealing. The results of SSA for this 10 city TSP with 100 different initial conditions are as follows [7]: the rate of global minima is 92 % and average iterations are 58700.

In this simulation, one iteration means one cyclic updating of all neuron states according to the Metropolis criterion and the temperature is decreased by one step after each iteration. Clearly, CSA is much faster and more efficient than this SSA for the 10 city TSP. In addition, we also applied TCNNs to the 48 USA city TSP and a electric power generator maintenance scheduling problem, which show the similar results as the 10 city TSP[7]. Thus, these simulation results have implied high searching ability of TCNNs. However, for practical applications to larger-scaled optimization problems, more sophisticated formulation combining TCNNs with powerful heuristic methods, such as the 2-opt algorithm, the tabu search and so on seems promising [18, 19].

9. Conclusion

We have reviewed our analyses on the nonlinear properties of neural networks with special emphasis on chaotic dynamics and the chaotic simulated annealing [6]-[11]. We have first provided the sufficient conditions for the asymptotical stability and local bifurcations of transiently chaotic neural networks (TCNNs), which are a generalized version of the discretized Amari-Hopfield neural networks, and then shown that there exists topologically chaotic structure with a snap-back repeller in a global attracting set. Furthermore, when certain conditions are satisfied, the attracting set eventually evolves into a strange attractor by a homoclinic bifurcation, which results in complicated chaotic dynamics. The existence of the strange attractor implies that TCNNs have a globally searching ability for globally optimal solutions of the commonly used objective functions in combinatorial optimization problems when the size of the strange attractor is sufficiently large to involve all minima. In addition, since chaos may be viewed as a deterministic noise, we develop a global optimization method, i.e., the chaotic simulated annealing, which has been verified having high ability to reach global minima with striking efficiency for various optimization problems by many experiments [23, 24, 26, 33, 34, 36, 37, 38, 39].

The theoretical results are useful for applications of TCNNs to practical problems, e.g., combinatorial optimization [7, 8, 10] and dynamical associative memory [1, 2], because they indicate that the attractor of the neural networks covers all the bounded regions including all minima of the commonly used objective function under some conditions [10]. From the computation viewpoint, these mathematical models of neural networks may also lead to significant improvement of computational efficiency.

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Existence and Unstable Sets of Oscillating Periodic Orbits for Delayed Excitatory Networks of Two Neurons

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Abstract

We study a system of delay differential equations which describes the dynamics of a delayed excitatory network of two neurons. Necessary and sufficient condition for the stability of the origin and the distribution of the eigenvalues of the linearized equation are obtained. We also obtain an oscillating periodic orbit in the closure of the global forward extension W of a 3-dimensional C^1-submanifold contained in the unstable set of the origin. Moreover, we completely describe the dynamics of the flow restricted to $\overline{W}$ and show that the unstable set of this periodic orbit is the boundary of W except the two nontrivial equilibria.

1. Introduction

Recently, Chen and Wu [2, 3] studied the following system of delay differential equations

$$\begin{cases} \dot{x}(t) = & -\mu x(t) + f(y(t-\tau)), \\ \dot{y}(t) = & -\mu y(t) + f(x(t-\tau)). \end{cases} \tag{1.1}$$

Using the theory of discrete Lyapunov functional developed by Mallet-Paret and Sell [7, 8] and the geometric approach of Krisztin, Walther, and Wu [6],

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they obtained some important properties of the global attractor of system (1.1) such as the global existence of periodic orbits and the unstable sets of the periodic orbits. Particularly, they got a spindle which is halved by a 2-dimensional closed disk and the disk is bordered by a phase-locked oscillating periodic orbit. Moreover, the boundary of the spindle except the two tips is the unstable set of this phase-locked periodic orbit.

The purpose of this paper is to generalize the results of Chen and Wu [2, 3] to a general system of system (1.1), i.e.,

$$\begin{cases} \dot{u}(t) = & -\mu_1 u(t) + f_1(v(t-\tau_1)), \\ \dot{v}(t) = & -\mu_2 v(t) + f_2(u(t-\tau_2)), \end{cases} \tag{1.2}$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$ with $\tau := \tau_1 + \tau_2 > 0$, and f_1 and f_2 from $\mathbb{R}$ to $\mathbb{R}$ are C^1-functions and satisfy

$$\begin{cases} f_1(0) = 0 = f_2(0), \\ f_1'(\xi) > 0 \text{ and } f_2'(\xi) > 0 \text{ for } \xi \in \mathbb{R}. \end{cases} \tag{H1}$$

System (1.2) describes the dynamics of a network of two saturatory neurons with delayed outputs, where the signs of the derivatives of the activation functions f_1 and f_2 indicate that the interactions between the two neurons are excitatory. For the detailed description of networks of neurons with delayed outputs, we refer to Hopfield [5], Marcus and Westervelt [9] and Wu [11].

To apply the discrete Lyapunov functional theory of cyclic systems developed in Mallet-Paret and Sell [7, 8], we first transform system (1.2) into

$$\begin{cases} \dot{x}(t) = & -\tau\mu_1 x(t) + \tau f_1(y(t)) \\ \dot{y}(t) = & -\tau\mu_2 y(t) + \tau f_2(x(t-1)) \end{cases} \tag{1.3}$$

by using the transformation

$$\begin{cases} x(t) = & u(\tau t), \\ y(t) = & v(\tau t - \tau_1). \end{cases}$$

Under hypothesis (H1), system (1.3) is a cyclic system of positive feedback in the sense of Mallet-Paret and Sell [7, 8].

Let $\mathbb{K} = [-1, 0] \cup \{1\}$ and $C(\mathbb{K}) = \{\varphi : \mathbb{K} \to \mathbb{R}; \varphi \text{ is continuous}\}$. Then $C(\mathbb{K})$ is a Banach space with the supernorm, which we choose as the phase space for system (1.3). Throughout this paper, we will always tacitly use the identification

$$C(\mathbb{K}) = C([-1, 0]; \mathbb{R}) \times \mathbb{R}$$

and write an element $\psi \in C(\mathbb{K})$ as $(\psi|_{[-1,0]}, \psi(1)) \in C([-1, 0]; \mathbb{R}) \times \mathbb{R}$.

Following Mallet-Paret and Sell [7], for each $\varphi \in C(\mathbb{K})$, there exists a unique pair of continuous mappings $x : [-1, \infty) \to \mathbb{R}$ and $y : [0, \infty) \to \mathbb{R}$ such that $(x, y) : (0, \infty) \to \mathbb{R}^2$ is continuously differentiable and satisfies (1.3) for $t > 0$, $x|_{[-1,0]} = \varphi|_{[-1,0]}$ and $y(0) = \varphi(1)$. Let $z^\varphi = (x^\varphi, y^\varphi)$ denote the above unique pair and define $z_t^\varphi = (x_t^\varphi, y^\varphi(t)) \in C(\mathbb{K})$ for $t \geq 0$, where $x_t^\varphi(\theta) = x^\varphi(t + \theta)$ for $\theta \in [-1, 0]$. Then the map $\Phi : \mathbb{R}^+ \times C(\mathbb{K}) \ni (t, \varphi) \mapsto z_t^\varphi \in C(\mathbb{K})$ is a continuous semiflow and Φ is monotone with respect to the partial ordering in $C(\mathbb{K})$ determined by the positive cone $K = \{\varphi \in C(\mathbb{K}); \varphi(\theta) \geq 0 \text{ for } \theta \in \mathbb{K}\}$ (see Chen and Wu [2]).

Due to the hypothesis (H1), one can easily show that, for any given $t \geq 0$, the map $\Phi(t, \cdot) : C(\mathbb{K}) \to C(\mathbb{K})$ is injective. In particular, for each $\varphi \in C(\mathbb{K})$ there exists at most one $z = (x, y) : \mathbb{R} \to \mathbb{R}^2$ which satisfies (1.3) and such that $z_0 = (x_0, y(0)) = \varphi$. If such a solution exists, it will be also denoted by $z^\varphi : \mathbb{R} \to \mathbb{R}^2$.

Our first step to the description of the global attractor of the semiflow Φ generated by system (1.3) is to show the existence of an oscillating periodic solution for the delay τ larger than a certain $\tau^* > 0$. We obtain the existence and the unstable set of such a periodic solution following the geometric approaches of Chen and Wu [2, 3]. Namely, we consider the global forward extension of a leading 3-dimensional unstable manifold tangent to the eigenspace of the generator of the C_0-semigroup $\{D_2\Phi(t, 0)\}_{t\geq 0}$ associated with the spectral set consisting of the unique real eigenvalue and a pair of conjugate eigenvalues with the largest positive real parts. To distinguish the oscillation frequencies, measured by the discrete Lyapunov functional, between the solutions in the aforementioned eigenspace and solutions in its complement, we need some detailed information about the eigenvalues of the generator of $\{D_2\Phi(t, 0)\}_{t\geq 0}$. We show, in Section 3, that there exists a $\tau^* > 0$ such that when $\tau > \tau^*$, the eigenvalues are simple and are given by a positive λ_0 and conjugate complex pairs $\{a_k \pm ib_k\}_{k\in\mathbb{N}}$ with $b_k \in (2(k-1)\pi, 2k\pi)$ for $k \in \mathbb{N}$. Moreover, $a_1 > 0$ and $\lambda_0 > a_1 > a_2 > \cdots > a_k > \cdots \to -\infty$ as $k \to \infty$.

Before finding the constant τ^*, we study the stability of the origin of system (1.3) in Section 2. We show that the necessary and sufficient condition for the stability of the origin of system (1.3) is $\mu_1\mu_2 > f_1'(0)f_2'(0)$. Then, under the hypothesis that $\mu_1\mu_2 < f_1'(0)f_2'(0)$, we obtain the constant τ^* and derive the aforementioned distribution of the eigenvalues of the infinitesimal generator of the C_0-semigroup $\{D_2\Phi(t, 0)\}_{t\geq 0}$ in Section 3.

The main results are obtained in Section 4, where we show that for $\tau > \tau^*$ there is a unique nontrivial oscillating periodic orbit in the closure of the global forward extension W of a 3-dimensional C^1-submanifold contained in the unstable set of the origin. We also completely describe the dynamics of the flow restricted to $\overline{W}$ and get the unstable set of this periodic orbit.

2. Local Stability of the Origin

The origin of system (1.3) is stable if and only if all the roots of

$$\lambda^2 + \tau p\lambda + \tau^2 q - \tau^2 r e^{-\lambda} = 0 \tag{2.1_τ}$$

lie to the left of the imaginary axis, where $p = \mu_1 + \mu_2$, $q = \mu_1\mu_2$, $r = f_1'(0)f_2'(0)$. Equation (2.1_τ) is the characteristic equation associated with the linearized system of system (1.3) at the origin.

Note that equation (2.1_τ) is equivalent to

$$(\lambda^2 + \tau p\lambda + \tau^2 q)e^{\lambda} - \tau^2 r = 0. \tag{2.2}$$

Since

$$(\tau p)^2 - 2(\tau^2 q) = \tau^2(\mu_1^2 + \mu_2^2) \geq 0$$

and

$$-\tau^2 r = -\tau^2 f_1'(0)f_2'(0) < 0$$

by hypothesis (H1), Theorem 13.9 of Bellman and Cooke [1] tells us that the necessary and sufficient condition that all roots of equation (2.2), and hence of (2.1_τ), lie to the left of the imaginary axis is

$$-\tau^2 q < -\tau^2 r < 0 \quad \text{and} \quad \frac{-\tau r \sin\alpha(\tau)}{p\alpha(\tau)} < 1, \tag{2.3}$$

where $\alpha(\tau) \in (\pi, 2\pi)$, which is unique, satisfies

$$\cot\alpha(\tau) = \frac{\alpha^2(\tau) - \tau^2 q}{\tau p\alpha(\tau)}. \tag{2.4}$$

Let $\alpha : (0, \infty) \ni \tau \mapsto \alpha(\tau) \in (\pi, 2\pi)$ be the function defined by (2.4). Then α is differentiable. Moreover,

Lemma 2.1. $\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = \frac{p\alpha(\tau^2 q+\alpha^2)}{\tau p(\tau^2 q+\alpha^2)+\tau^2 p^2\alpha^2+(\alpha^2-\tau^2 q)^2}.$

Proof. Equation (2.4) can be rewritten as

$$\alpha^2(\tau) - \tau^2 q = \tau p \alpha(\tau) \cot \alpha(\tau). \tag{2.5}$$

Differentiating equation (2.5) with respect to τ and rearranging the terms, we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = \frac{2\tau q + p\alpha \cot \alpha}{2\alpha - \tau p \cot \alpha + \frac{\tau p \alpha}{\sin^2 \alpha}},$$

which, combined with $\frac{1}{\sin^2 \alpha} = 1 + \cot^2 \alpha$ and (2.4), gives us

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = \frac{p\alpha(\tau^2 q + \alpha^2)}{\tau p(\tau^2 q + \alpha^2) + \tau^2 p^2 \alpha^2 + (\alpha^2 - \tau^2 q)^2}.$$

This completes the proof.

Proposition 2.2. The function $h : (0, \infty) \ni \tau \mapsto \frac{-\tau r \sin \alpha(\tau)}{p\alpha(\tau)} \in (0, \infty)$ is a strictly increasing function.

Proof. Using Lemma 2.1 and equation (2.4), we have

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}\tau} &= -\frac{p\alpha(r \sin \alpha + \tau r \cos \alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}) - \tau p r \sin \alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}}{p^2 \alpha^2} \\
&= -\frac{r p \alpha \sin \alpha + r(\alpha^2 - \tau^2 q) \sin \alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau} - \tau p r \sin \alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}}{p^2 \alpha^2} \\
&= -\frac{r \sin \alpha}{p^2 \alpha^2} \left[p\alpha + (\alpha^2 - \tau^2 q - \tau p) \frac{\mathrm{d}\alpha}{\mathrm{d}\tau} \right] \\
&= -\frac{r \sin \alpha}{p^2 \alpha^2} \left[p\alpha + \frac{p\alpha(\tau^2 q + \alpha^2)(\alpha^2 - \tau^2 q - \tau p)}{\tau p(\tau^2 q + \alpha^2) + \tau^2 p^2 \alpha^2 + (\alpha^2 - \tau^2 q)^2} \right] \\
&= -\frac{r \sin \alpha (2\alpha^4 + \tau^2 \alpha^2 (p^2 - 2q))}{p\alpha[\tau p(\tau^2 q + \alpha^2) + \tau^2 p^2 \alpha^2 + (\alpha^2 - \tau^2 q)^2]} \\
&> 0
\end{aligned}$$

since $p^2 - 2q = \mu_1^2 + \mu_2^2 > 0$ and $\alpha \in (\pi, 2\pi)$. Therefore, h is strictly increasing on $(0, \infty)$ and the proof is complete.

Since $\alpha(\tau) \in (\pi, 2\pi)$, by using (2.4), we get

$$\begin{aligned}
\lim_{\tau \to \infty} h(\tau) &= \lim_{\tau \to \infty} \frac{\tau r}{p\alpha \sqrt{1 + \cot^2 \alpha}} \\
&= \lim_{\tau \to \infty} \frac{\tau^2 r}{\sqrt{\tau^2 p^2 \alpha^2 + (\alpha^2 - \tau^2 q)^2}} \\
&= \lim_{\tau \to \infty} \frac{\tau^2 r}{\sqrt{\alpha^4 + \tau^2 \alpha^2 (\mu_1^2 + \mu_2^2) + \tau^4 q^2}} \\
&= \frac{r}{q}.
\end{aligned} \tag{2.6}$$

Observe that $-\tau^2 q < -\tau^2 r$ implies $r < q$. Hence if $r < q$, it follows from (2.6) and Proposition 2.2 that $h(\tau) < 1$ for all τ. Combining this with the discussion in the beginning of this section, we have proved the following theorem.

Theorem 2.3. The origin of system (1.3) is (locally) asymptotically stable if and only if $f_1'(0)f_2'(0) < \mu_1\mu_2$.

3. Eigenvalues of the Generator

Before studying the periodic solutions of system (1.3), we need some information about the eigenvalues of the infinitesimal generator of the C_0-semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$.

It is well known that the eigenvalues of the generator of $\{D_2\Phi(t,0)\}_{t\geq 0}$ coincide with the roots of equation (2.1_τ). Here, we want to show that roots of (2.1_τ) in $\mathbb{C}_+ = \{\lambda \in \mathbb{C}; \operatorname{Im}\lambda \geq 0\}$ are distributed in a union of strips S_k (defined later) and $\mathbb{R}$, and the real parts of these roots are well ordered. Results of this nature are expected in Mallet-Paret and Sell [7] (Theorem 3.2 and Corollary 3.3), and our discussions in this section seem to confirm their expectation.

By Theorem 2.3, the origin of system (1.3) is stable if $\mu_1\mu_2 > f_1'(0)f_2'(0)$. Hence, from now on, we also assume that

$$\mu_1\mu_2 < f_1'(0)f_2'(0). \tag{H2}$$

Under hypothesis (H2), it follows from (2.6) that $\lim_{\tau\to\infty} h(\tau) > 1$. Note that $\lim_{\tau\to 0^+} h(\tau) = 0$ since $\alpha(\tau) \in (\pi, 2\pi)$. Thus, by Proposition 2.2, there exists a unique τ^* such that

$$h(\tau) \begin{cases} > 1 & \text{if } \tau > \tau^*, \\ = 1 & \text{if } \tau = \tau^*, \\ < 1 & \text{if } \tau < \tau^*. \end{cases}$$

Moreover, we can easily check that $i\alpha(\tau^*)$ is a root of (2.1_{τ^*}).

Observe that λ is a root of (2.1_τ) if and only if its complex conjugate $\overline{\lambda}$ is. Therefore, we can restrict our study of the distribution of the roots of equation (2.1_τ) to the upper semi-plane $\mathbb{C}_+$.

Recall that equation (2.1_τ) can be rewritten as

$$g(\lambda) = (\lambda^2 + \tau p\lambda + \tau^2 q)e^{\lambda} - \tau^2 r = 0. \tag{3.1}$$

Then, $g(0) = \tau^2 q - \tau^2 r < 0$ by hypothesis (H2). Note that $g'(\lambda) = (\lambda^2 + (\tau p + 2)\lambda + \tau^2 q + \tau p)e^{\lambda}$, which is positive for $\lambda \geq 0$, and $g(\lambda) \to \infty$ as $\lambda \to \infty$. Therefore, equation (3.1), and hence equation (2.1_τ), always has a unique nonnegative real root, say λ_0. Moreover, $\lambda_0 > 0$ and λ_0 is simple. Also, we can see that $g(\lambda)$ attains its local maximum at $\lambda_{\max}(\tau) = -\frac{\tau p + 2 + \sqrt{4 + \tau^2(\mu_1 - \mu_2)^2}}{2}$ and the local maximum is

$$k(\tau) = (2 + \sqrt{4 + \tau^2(\mu_1 - \mu_2)^2})e^{-\frac{\tau p + 2 + \sqrt{4 + \tau^2(\mu_1 - \mu_2)^2}}{2}} - \tau^2 r.$$

Observe that $k(0) = 4e^{-2} > 0$, $k(\tau) \to -\infty$ as $\tau \to \infty$ and $\frac{\mathrm{d}k}{\mathrm{d}\tau} < 0$. Thus, there exists a unique $\tau_* > 0$ such that

$$k(\tau) \begin{cases} < 0 & \text{if } \tau > \tau_*, \\ = 0 & \text{if } \tau = \tau_*, \\ > 0 & \text{if } \tau < \tau_*. \end{cases}$$

The following result follows directly.

Proposition 3.1. If $\tau > \tau_*$, then equation (2.1_τ) has a unique simple real root λ_0 and $\lambda_0 > 0$. If $\tau = \tau_*$, then equation (2.1_τ) has only two real roots λ_0 and $\lambda_{\max}(\tau_*)$, where λ_0 is simple and $\lambda_{\max}(\tau_*)$ is of multiplicity 2 and $\lambda_{\max}(\tau_*) < 0 < \lambda_0$. If $\tau < \tau_*$, then equation (2.1_τ) has only three simple real roots λ_0, λ_1 and λ_1', and $\lambda_1' < \lambda_1 < 0 < \lambda_0$.

Proposition 3.2. All roots of equation (2.1_τ) except $\lambda_{\max}(\tau_*)$ are simple.

Proof. Let ν be a root of (2.1_τ) with multiplicity larger than 1. Then ν satisfies both

$$\nu^2 + \tau p \nu + \tau^2 q - \tau^2 r e^{-\nu} = 0$$

and

$$2\nu + \tau p + \tau^2 r e^{-\nu} = 0.$$

Thus ν satisfies

$$\nu^2 + (\tau p + 2)\nu + \tau^2 q + \tau p = 0.$$

Noting that $(\tau p + 2)^2 - 4(\tau^2 q + \tau p) = \tau^2(\mu_1 - \mu_2)^2 + 4 > 0$, we know that ν must be negative. Hence, $\nu = \lambda_{\max}(\tau_*)$ by Proposition 3.1. This completes the proof.

Lemma 3.3. There is no root of (2.1_τ) in line $a + bi$, where $b = 2k\pi$ for some $k \in \mathbb{N}$.

Proof. By way of contradiction, assume that there exist a $k \in \mathbb{N}$ and $\tau > 0$ such that $a + 2k\pi i$ is a root of (2.1_τ). Then, separating the real and imaginary parts, we have

$$\begin{cases} a^2 - 4k^2\pi^2 + \tau pa + \tau^2 q - \tau^2 r e^{-a} = 0 \\ 4ka\pi + 2\tau pk\pi = 0 \end{cases}$$

and this system is not compatible. In fact, from the second equation, we have $a = -\frac{\tau p}{2}$. Substituting it into the left hand side of the first equation, we get $-4k^2\pi^2 - \frac{\tau^2 p^2}{4} + \tau^2 q - \tau^2 r e^{\frac{\tau p}{2}} = -4k^2\pi^2 - \frac{\tau^2(\mu_1 - \mu_2)^2}{4} - \tau^2 r e^{\frac{\tau p}{2}} < 0$, a contradiction. This completes the proof.

For $k \in \mathbb{N}$, define

$$\begin{aligned} S_k &= \{\lambda \in \mathbb{C}; 2(k-1)\pi < \operatorname{Im}\lambda < 2k\pi\}, \\ \alpha_k &= \alpha(\tau^*) + 2(k-1)\pi, \\ \tau_k &= \frac{\tau^*\alpha_k}{\alpha_1}. \end{aligned}$$

One can show that $i\alpha_k \in S_k$ is a solution of (2.1_{τ_k}). Moreover Proposition 3.1 and Lemma 3.3 combined assert that there exists a subset $A \subseteq \mathbb{N}$ such that all roots of (2.1_τ) in $\mathbb{C}_+$ are in the union of strips S_k for $k \in A$ and $\mathbb{R}$.

To continue our discussion, we need a result about the continuous dependence of the roots of (2.1_τ) on τ. Since this result will be used repeatedly, we state it for the sake of easy reference.

Theorem 3.4 (Lemma XI.2.8 of [4]). Let Ω be an open set in $\mathbb{C}$. Let ω be an open subset of Ω whose closure $\overline{\omega}$ is compact and contained in Ω. Let $\bar{\tau}$ be such that no root of $(2.1_{\bar{\tau}})$ is on the boundary of ω. Then there exists a neighborhood U of $\bar{\tau}$ in $\mathbb{R}$ such that

(i) for any $\tau \in U$, equation (2.1_τ) has no zeros on the boundary of ω;

(ii) the number of zeros of (2.1_τ) in ω, taking multiplicities into account, is constant for $\tau \in U$.

Lemma 3.5. For $k \in N$ and $k \geq 2$, equation (2.1_τ) has at least one root in S_k.

Proof. Let

$$A_k = \{\tau > 0; (2.1_\tau) \text{ has at least one root in } S_k\}.$$

Then $A_k \neq \emptyset$ since $\tau_k \in A_k$. It follows from Theorem 3.4 that A_k is relatively open in $(0, \infty)$. Since $(0, \infty)$ is connected, the proof is completed by showing that A_k is also relatively closed in $(0, \infty)$. Let $\{s_n\}_{n=1}^{\infty} \subseteq A_k$ be a sequence such that $s_n \to s_0$ as $n \to \infty$ for some $s_0 \in (0, \infty)$. For each $n \in \mathbb{N}$, let $a_n + ib_n \in S_k$ be a root of (2.1_{s_n}). It follows from (2.1_τ) that $\{a_n + ib_n\}_{n=1}^{\infty}$ is bounded. Therefore, there exists a subsequence of $\{a_n + ib_n\}$, say itself for the convenience of notation, and $a_0 + ib_0$ with $b_0 \in [2(k-1)\pi, 2k\pi]$ such that $a_n + ib_n \to a_0 + ib_0$ as $n \to \infty$. Taking limit in (2.1_{s_n}), we know that $a_0 + ib_0$ is a root of (2.1_{s_0}). It follows from Lemma 3.3 that $a_0 + ib_0 \in S_k$ and hence $s_0 \in A_k$. This completes the proof.

Lemma 3.6. Let $A_1 = \{\tau > 0; (2.1_\tau) \text{ has at least one root in } S_1\}$. Then $A_1 = (\tau_*, \infty)$ and hence $\tau_* < \tau^*$.

Proof. Let $\hat{\tau} = \inf A_1$. We first show that $\hat{\tau} = \tau_*$. Choose a decreasing sequence $\{s_n\}_{n=1}^{\infty} \subseteq A_1$ such that $s_n \to \hat{\tau}$ as $n \to \infty$. Let $a_n + ib_n$ and $c_n > 0$ be the roots of (2.1_{s_n}) in S_1 and $\mathbb{R}$, respectively. Then it follows from (2.1_τ) that $\{a_n + ib_n, c_n\}_{n=1}^{\infty}$ is bounded. Therefore, there exist subsequences of $\{a_n + ib_n\}$ and $\{c_n\}$, say themselves for the convenience of notations, and $a_0 + ib_0$ with $b_0 \in [0, 2\pi]$ and $c_0 \geq 0$ such that $a_n + ib_n \to a_0 + ib_0$ and $c_n \to c_0$ as $n \to \infty$, respectively. Taking limit in (2.1_{s_n}), we know that $a_0 + ib_0$ and c_0 are roots of $(2.1_{\hat{\tau}})$. Moreover, by the definition of $\hat{\tau}$ and Lemma 3.3 and Theorem 3.4, we know that $b_0 = 0$. We claim that $\hat{\tau} \neq 0$. If $\hat{\tau} = 0$, then $(2.1_{\hat{\tau}})$ reduces to $\lambda^2 = 0$. So the only root is 0 which is of multiplicity 2. But the fact that $c_n \to c_0$ and $a_n \pm ib_n \to a_0 \pm ib_0$ as $n \to \infty$ implies that 0 is a root of multiplicity larger than 2, a contradiction. This proves the claim. Now, again, we know that a_0 is a multiple root of $(2.1_{\hat{\tau}})$. It follows from Proposition 3.1 that $\hat{\tau} = \tau_*$. Moreover, since $\tau^* \in A_1$, we have $\tau_* < \tau^*$.

Now, we show that $A_1 = (\tau_*, \infty)$. It follows from $\tau_* = \inf A_1$ that $A_1 \subseteq (\tau_*, \infty)$. By Theorem 3.4, we know that A_1 is relatively open in (τ_*, ∞). Since (τ_*, ∞) is connected, the proof is completed by showing that A_1 is also relatively closed in (τ_*, ∞). Let $\{t_m\}_{m=1}^{\infty} \subseteq A_1$ be a sequence such that $t_m \to t_0$ as $m \to \infty$ for some $t_0 > \tau_*$. For each $m \in \mathbb{N}$, let $a_m + ib_m \in S_1$ be a root of (2.1_{t_m}). It follows from (2.1_τ) that $\{a_m + ib_m\}_{m=1}^{\infty}$ is bounded. Therefore, there exist a subsequence of $\{a_m + ib_m\}$, say itself for the convenience of notation, and $a_0 + ib_0$ with $b_0 \in [0, 2\pi]$ such that $a_m + ib_m \to a_0 + ib_0$ as $m \to \infty$. Taking limit in (2.1_{t_m}), we know that $a_0 + ib_0$ is a root of (2.1_{t_0}). We claim that $b_0 \neq 0$. By way of contradiction, assume that $b_0 = 0$. Noting that $a_m \pm ib_m$ are roots of (2.1_{t_m}) and $a_m \pm ib_m \to a_0$, we know that a_0 is a multiple root of (2.1_{t_0}). It follows from Proposition 3.1 that $t_0 = \tau_*$, a contradiction to $t_0 > \tau_*$. This proves the claim. Moreover, it follows from Lemma 3.3 that $b_0 \neq 2\pi$. Therefore, $a_0 + ib_0 \in S_1$ and hence $t_0 \in A_1$, *i.e.*, A_1 is relatively closed in (τ_*, ∞). This completes the proof.

Lemma 3.7. For any $k \in \mathbb{N}$, let $a_k(\tau) + ib_k(\tau) \in S_k$ be a root of (2.1_τ). Then $a_k(\tau) < (>)\, 0$ if $\tau < (>) \tau_k$.

Proof. We only prove the case where $\tau < \tau_1 = \tau^*$. The proofs of the other cases are quite similar but simple and hence are omitted.

By way of contradiction, assume that there exist a $\bar{\tau} < \tau_1$ and a solution $a_1(\bar{\tau}) + ib_1(\bar{\tau}) \in S_1$ of $(2.1_{\bar{\tau}})$ such that $a_1(\bar{\tau}) \geq 0$. It follows from Theorem 3.4 and Lemma 3.6 that on (τ_*, ∞) there exists a differentiable solution curve of (2.1_τ), say $\gamma(\tau) + i\beta(\tau) \in S_1$, passing through $a_1(\bar{\tau}) + ib_1(\bar{\tau})$. Let

$$B_1 = \{\tau < \tau_1; \gamma(\tau) > 0\}.$$

If $a_1(\bar{\tau}) > 0$ then $\bar{\tau} \in B_1$. If $a_1(\bar{\tau}) = 0$ then $b_1(\bar{\tau}) \neq 0$ since 0 is not a root of $(2.1_{\bar{\tau}})$. It follows from (2.1_τ) that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}(\bar{\tau}) = \frac{2[b_1(\bar{\tau})]^4 + \bar{\tau}^2[b_1(\bar{\tau})]^2[\mu_1^2 + \mu_2^2]}{\bar{\tau}[(\bar{\tau}p - ([b_1(\bar{\tau})]^2 + \bar{\tau}^2 q))^2 + (2b_1(\bar{\tau}) + \bar{\tau}pb_1(\bar{\tau}))^2]} > 0.$$

Thus, there is a $\tilde{\tau} \in (\bar{\tau}, \tau_1)$ such that $\tilde{\tau} \in B_1$. Therefore, $B_1 \neq \emptyset$. Let $\hat{\tau} = \sup B_1$, which is less than or equal to τ_1. Then either (i) $\gamma(\hat{\tau}) = 0$ or (ii) $\beta(\hat{\tau}) = 0$ and $\gamma(\hat{\tau}) > 0$ by Theorem 3.4 and Lemma 3.3. If $\gamma(\hat{\tau}) = 0$ then $\beta(\hat{\tau}) \neq 0$ since 0 is not a root of $(2.1_{\hat{\tau}})$. Hence $\hat{\tau} \notin B_1$. Again, we have $\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}(\hat{\tau}) > 0$, which implies that $\gamma(\tau) < 0$ for all $\tau < \hat{\tau}$ but close enough to $\hat{\tau}$, a contradiction to the choice of $\hat{\tau}$. If $\beta(\hat{\tau}) = 0$ and $\gamma(\hat{\tau}) > 0$, by Theorem 3.4 and the fact that $\gamma(\tau) \pm i\beta(\tau)$ are roots of (2.1_τ), then $\gamma(\hat{\tau}) > 0$ is a multiple root of $(2.1_{\hat{\tau}})$, a contradiction to Proposition 3.1. This completes the proof.

Corollary 3.8. For any $k \in \mathbb{N}$, $i\alpha_k \in S_k$ is the unique solution of (2.1_{τ_k}) in S_k.

Proof. If $a_k + ib_k \in S_k$ is a solution of (2.1_{τ_k}), then $a_k = 0$ by Theorem 3.4 and Lemma 3.7. Recall that $i\alpha_k \in S_k$ is a root of (2.1_{τ_k}). Note that if $ib \in S_k$ is a solution of (2.1_{τ_k}), then it satisfies $((ib)^2 + i\tau_k pb + \tau_k^2 q)e^{ib} - \tau_k^2 r = 0$. Separating the real and imaginary parts, we have

$$(\tau_k^2 q - b^2)\cos b - \tau_k pb \sin b - \tau_k^2 r = 0 \tag{3.2}$$

and

$$(\tau_k^2 q - b^2)\sin b + \tau_k pb \cos b = 0. \tag{3.3}$$

From (3.3), we have $\cot b = \frac{b^2 - \tau_k^2 q}{\tau_k pb}$ and substitute it into (3.2) to obtain

$$-\tau_k^2 r - \frac{\sin b}{\tau_k pb}[(b^2 - \tau_k^2 q)^2 + \tau_k^2 p^2 b^2] = 0.$$

Thus, $b \in ((2k-1)\pi, 2k\pi)$. Let $\nu > 0$ be such that $b = \alpha_k \nu$. Then $\nu \in (\frac{2k-1}{\alpha_k}\pi, \frac{2k\pi}{\alpha_k})$ and

$$\cot(\alpha_k \nu) = \frac{\alpha_k^2 \nu^2 - \tau_k^2 q}{\tau_k p \alpha_k \nu}. \tag{3.4}$$

Let $H : (\frac{2k-1}{\alpha_k}\pi, \frac{2k\pi}{\alpha_k}) \to \mathbb{R}$ be defined by

$$H(\nu) = \cot(\alpha_k \nu) - \frac{\alpha_k^2 \nu^2 - \tau_k^2 q}{\tau_k p \alpha_k \nu}.$$

Then $H(1) = 0$ and

$$H'(\nu) = -\frac{\alpha_k}{\sin^2(\alpha_k \nu)} - \frac{\tau_k p \alpha_k^3 \nu + \tau_k^3 pq\alpha_k}{(\tau_k p \alpha_k \nu)^2} < 0.$$

Thus equation (3.4) has one unique root $\nu = 1$, i.e., $b = \alpha_k$. This completes the proof.

It follows from Lemma 3.7 and Corollary 3.8 that equation (2.1_τ) has pure imaginary roots only when $\tau = \tau_k$ for some $k \in \mathbb{N}$ and the roots are $\pm i\alpha_k$.

Now, we are ready to state the main result of this section, which plays an important role in the next section.

Theorem 3.9. For $k \in \mathbb{N}$ with $k \geq 2$, equation (2.1_τ) has a unique simple solution in S_k, denoted by $a_k(\tau) + ib_k(\tau)$, where $a_{k+1} < a_k$ and hence $a_k \to -\infty$ as $k \to \infty$. Moreover, if $\tau > \tau_*$, equation (2.1_τ) has a unique simple solution in S_1, denoted by $a_1(\tau) + ib_1(\tau)$, and $\{a_k \pm ib_k\}_{k\geq 1} \cup \{\lambda_0\}$ are all solutions of (2.1_τ) and $a_2 < a_1 < \lambda_0$; if $\tau = \tau_*$, $\{a_k \pm ib_k\}_{k\geq 2} \cup \{\lambda_{\max}(\tau_*), \lambda_0\}$ are all solutions of (2.1_τ) and $a_2(\tau_*) < \lambda_{\max}(\tau_*) < 0 < \lambda_0$; if $\tau < \tau_*$, $\{a_k \pm ib_k\}_{k\geq 2} \cup \{\lambda_1', \lambda_1, \lambda_0\}$ are all solutions of (2.1_τ) and $a_2 < \lambda_1' < \lambda_1 < 0 < \lambda_0$.

Proof. The assertions of existence and uniqueness in the theorem follow easily from Proposition 3.1, Proposition 3.2, Lemma 3.3, Theorem 3.4, Lemma 3.5, Lemma 3.6 and Corollary 3.8. We only need to establish the inequalities. If this is true, then $\lim\limits_{k\to\infty} a_k$ exists and it follows from (2.1_τ) that $\lim\limits_{k\to\infty} a_k = -\infty$.

Note that $\lambda_{\max}(\tau_*) < 0 < \lambda_0$ and $\lambda_1' < \lambda_1 < 0 < \lambda_0$ have been established in Proposition 3.1. To complete the proof, we distinct four cases.

Case 1. $a_{k+1} < a_k$ for $k \in \mathbb{N}$ and $k \geq 2$. Let

$$C_k = \{\tau > 0; a_{k+1}(\tau) < a_k(\tau)\}.$$

By Lemma 3.7, $a_{k+1}(\tau) < 0 < a_k(\tau)$ if $\tau \in (\tau_k, \tau_{k+1})$. Thus $C_k \neq \emptyset$. It follows from Theorem 3.4 that C_k is relatively open in $(0,\infty)$. Since $(0,\infty)$ is connected, $C_k = (0,\infty)$ if C_k is also relatively closed in $(0,\infty)$. Now, we show it. Let $\{s_n\}_{n=1}^{\infty} \subseteq C_k$ be a sequence such that $s_n \to s_0$ as $n \to \infty$ for some $s_0 > 0$. If $s_0 \notin C_k$, then $a_{k+1}(s_0) = a_k(s_0)$. However, it follows from (2.1_τ) that

$$\begin{aligned}
& s_0^4 r^2 e^{-2a_k(s_0)} \\
= \;& a_k^4(s_0) + b_k^4(s_0) + s_0^2 p^2 a_k^2(s_0) + s_0^4 q^2 + 2s_0 p a_k^3(s_0) + 2s_0^2 a_k^2(s_0) q \\
& + s_0^2 b_k^2(s_0)(\mu_1^2 + \mu_2^2) + 2s_0^3 pq a_k(s_0) \\
& + 2s_0 p a_k(s_0) b_k^2(s_0) + 2a_k^2(s_0) b_k^2(s_0) \\
= \;& a_{k+1}^4(s_0) + b_{k+1}^4(s_0) + s_0^2 p^2 a_{k+1}^2(s_0) + s_0^4 q^2 + 2s_0^2 p a_{k+1}^3(s_0) \\
& + 2s_0 a_{k+1}^2(s_0) q + s_0^2 b_{k+1}^2(s_0)(\mu_1^2 + \mu_2^2) + 2s_0^3 pq a_{k+1}(s_0) \\
& + 2s_0 p a_{k+1}(s_0) b_{k+1}^2(s_0) + 2a_{k+1}^2(s_0) b_{k+1}^2(s_0) \\
= \;& s_0^4 r^2 e^{-2a_{k+1}(s_0)},
\end{aligned}$$

which gives us

$$\begin{aligned}
0 \;=\; & [b_k^2(s_0) - b_{k+1}^2(s_0)][b_k^2(s_0) + b_{k+1}^2(s_0) + s_0^2(\mu_1^2 + \mu_2^2) \\
& + 2s_0 p a_k(s_0) + 2a_k^2(s_0)]
\end{aligned}$$

$$= [b_k^2(s_0) - b_{k+1}^2(s_0)][b_k^2(s_0) + b_{k+1}^2(s_0) + (s_0\mu_1 + a_k(s_0))^2 + (s_0\mu_2 + a_k(s_0))^2].$$

This is impossible since $a_k(s_0)+ib_k(s_0) \in S_k$ and $a_{k+1}(s_0)+ib_{k+1}(s_0) \in S_{k+1}$.

Case 2. $a_2 < a_1$ and $a_1 < \lambda_0$ when $\tau > \tau_*$. The proofs are similar as that in case 1 and hence are omitted.

Case 3. $a_2(\tau_*) < \lambda_{\max}(\tau_*)$ when $\tau = \tau_*$. First, similar argument as that at the end of Case 1 gives $a_2(\tau_*) \neq \lambda_{\max}(\tau_*)$. By the uniqueness and Theorem 3.4, we know that $\lim_{\tau \to \tau_*^+} a_2(\tau) = a_2(\tau_*)$ and $\lim_{\tau \to \tau_*^+} a_1(\tau) = \lambda_{\max}(\tau_*)$. This, combined with the result of Case 2 and $a_2(\tau_*) \neq \lambda_{\max}(\tau_*)$, implies that $a_2(\tau_*) < \lambda_{\max}(\tau_*)$.

Case 4. $a_2 < \lambda_1'$ when $\tau < \tau_*$. Again, similar argument as that at the end of Case 1 gives $a_2(\tau) \neq \lambda_1'(\tau)$ for $\tau < \tau_*$. By uniqueness and Theorem 3.4, we know that $\lim_{\tau \to \tau_*^-} a_2(\tau) = a_2(\tau_*)$ and $\lim_{\tau \to \tau_*^-} \lambda_1'(\tau) = \lambda_{\max}(\tau_*)$. Using the result of Case 3, we find a $\bar{\tau} < \tau_*$ such that $a_2(\bar{\tau}) < \lambda_1'(\bar{\tau})$. Now, we claim that $a_2(\tau) < \lambda_1'(\tau)$ for all $\tau < \tau_*$. By way of contradiction, assume that there exists a $\hat{\tau} < \tau_*$ such that $a_2(\hat{\tau}) \geq \lambda_1'(\hat{\tau})$. Then $a_2(\hat{\tau}) > \lambda_1'(\hat{\tau})$ since $a_2(\hat{\tau}) \neq \lambda_1'(\hat{\tau})$. However, by Theorem 3.4, there exists a $\tilde{\tau}$ between $\bar{\tau}$ and $\hat{\tau}$ such that $a_2(\tilde{\tau}) = \lambda_1'(\tilde{\tau})$, a contradiction to $a_2(\tilde{\tau}) \neq \lambda_1'(\tilde{\tau})$. Therefore, the claim is proved and this completes the proof.

4. Existence and Unstable Set of Oscillating Periodic Orbit

Remember that in this paper we focus on periodic solutions of system (1.3). It follows from Theorem 2.3 and Theorem 3.9 that, in order to obtain periodic solutions of system (1.3) as a Hopf bifurcation from the origin, it is necessary to assume that

$$\tau > \tau^*. \tag{H3}$$

From now on, we always assume that (H1)–(H3) hold.

Recall that the eigenvalues of the generator of the C_0-semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$ coincide with the roots of (2.1_τ). It follows from Theorem 3.9 and Lemma 3.7 that all the eigenvalues are simple and given by $\{a_k \pm ib_k\}_{k\geq 1} \cup \{\lambda_0\}$, where $\lambda_0 > a_1 > 0$ and $a_1 > a_2 > \cdots > a_k > \cdots \to -\infty$ as $k \to \infty$ and $b_k \in (2(k-1)\pi, 2k\pi)$ for $k \geq 1$.

Besides hypotheses (H1), (H2) and (H3), we also need the following hypothesis.

$$\text{There exists } M > 0 \text{ such that } \frac{f_1\left(\frac{f_2(\xi)}{\mu_2}\right)}{\xi} < \mu_1 \text{ if } |\xi| \geq M. \tag{H4}$$

It follows from (H1), (H2) and (H4) that there are $\xi^- < 0 < \xi^+$ so that $\frac{f_1(\frac{f_2(\xi^\pm)}{\mu_2})}{\xi^\pm} = \mu_1$ and $\frac{f_1(\frac{f_2(\xi)}{\mu_2})}{\xi} > \mu_1$ for $\xi \in (\xi^-, \xi^+) \setminus \{0\}$. Moreover, it is easy to see that $(\xi^\pm, \frac{f_2(\xi^\pm)}{\mu_2})$ are equilibria of system (1.3).

In what follows, we only state the main results of existence and unstable set of a nontrivial oscillating periodic orbit for system (1.3). Same and similar results as those in Chen and Wu [2, 3] can be stated and the proofs are almost the same with minor modifications only. Thus the proofs are omitted here because of the limitation on space.

Let P_0, P_1 and Q denote the realified generalized eigenspaces of the generator of the semigroup $\{D_2\Phi(t,0)\}_{t\geq 0}$ on $C(\mathbb{K})$ associated with the spectral sets $\{\lambda_0\}$, $\{a_1 \pm ib_1\}$ and $\{a_k \pm ib_k\}_{k\geq 2}$, respectively. Then

$$C(\mathbb{K}) = P_0 \oplus P_1 \oplus Q.$$

Choose $\beta > 1$ such that $\beta \in (e^{a_2}, e^{a_1})$. Then we can find bounded open neighborhoods N_Q, N_1 and N_0 of 0 in Q, P_1 and P_0 respectively, and a C^1-map $w_{loc} : N_1 \oplus N_0 \to Q$ with $w_{loc}(0) = 0$, $Dw_{loc}(0) = 0$, $w_{loc}(N_1 + N_0) \subseteq N_Q$ and so that the graph $W_{loc} = \{\chi + w_{loc}(\chi); \chi \in N_1 + N_0\}$ coincides with the set of $\varphi \in N_{loc} := N_0 + N_1 + N_Q$ such that there is a sequence $\{\varphi_n\}_{n=-\infty}^{0}$ with $\varphi_0 = \varphi$, $\varphi_n = \Phi(1, \varphi_{n-1})$ and $\varphi_n\beta^{-n} \in N_{loc}$ for $n \leq 0$, and $\varphi_n\beta^{-n} \to 0$ as $n \to -\infty$.

Let

$$W = \Phi(\mathbb{R}^+ \times W_{loc})$$

be the forward extension of W_{loc}. It is easy to show that for every $\varphi \in W$ there exists one and only one C^1-map $z^\varphi = (x^\varphi, y^\varphi) : \mathbb{R} \to \mathbb{R}^2$ with $z_0^\varphi = \varphi$ and satisfies system (1.3) for all $t \in \mathbb{R}$. Moreover, $z_t^\varphi \in W$ for all $t \in \mathbb{R}$ and $z_t^\varphi \to 0$ as $t \to -\infty$. Note that W is independent of the choice of β.

Now, we define the separatrix S by

$$S = \{\varphi \in C(\mathbb{K}); \varphi = 0 \text{ or } V(z_t^\varphi) > 0 \text{ for } t \geq 0\},$$

where $V : C(\mathbb{K}) \setminus \{0\} \to \{0, 2, 4, \ldots\}$ is the discrete Lyapunov functional introduced by Mallet-Paret and Sell [7].

Theorem 4.1. (i) There exists a unique nontrivial oscillating periodic orbit, say $\mathcal{O}$, for system (1.3) in $\overline{W}$. In fact, $\mathcal{O}$ is the boundary of $\overline{W} \cap S$, which is homeomorphic to a solid disk. Moreover, if $(p^1, p^2) : \mathbb{R} \to \mathbb{R}^2$ is a representative of $\mathcal{O}$ then the minimal period of $\mathcal{O}$, which is larger than 1, is determined by three consecutive zeros of p^1 or p^2.
(ii) $z_t^{\varphi} \to \mathcal{O}$ as $t \to \infty$ and $z_t^{\varphi} \to 0$ as $t \to -\infty$ for all $\varphi \in W \cap S \setminus \{0\}$.
(iii) $\overline{W} \cap S$ divides $\overline{W}$ into two halves. One is said to be above and the other below S. For $\varphi \in W \setminus S$, if φ is above (resp. below) S then $z_t^{\varphi} \to \xi_+$ (resp. ξ_-) as $t \to \infty$ and $z_t^{\varphi} \to 0$ as $t \to -\infty$. For $\varphi \in bd\,W \setminus (\mathcal{O} \cup \{\xi_-, \xi_+\})$, where $bd\,W = \overline{W} \setminus W$, if φ is above (resp. below) S then $z_t^{\varphi} \to \xi_+$ (resp. ξ_-) as $t \to \infty$ and $z_t^{\varphi} \to \mathcal{O}$ as $t \to -\infty$. Here, $\xi_{\pm} = (\xi^{\pm}_{[-1,0]}, \frac{f_2(\xi^{\pm})}{\mu_2}) \in C(\mathbb{K})$, where $\xi^{\pm}_{[-1,0]}(\theta) = \xi^{\pm}$ for $\theta \in [-1, 0]$.

The stability of the periodic orbit $\mathcal{O}$ obtained in Theorem 4.1 is determined by its Floquet multipliers. It is well known that the Floquet multipliers are eigenvalues of finite multiplicity forming the spectrum away from 0 of the compact monodromy operator $M = D_2\Phi(\omega, p_0)$, where ω is the minimal period of $\mathcal{O}$ and $p_0 = (p_0^1, p^2(0)) \in C(\mathbb{K})$ with $(p^1, p^2) : \mathbb{R} \to \mathbb{R}^2$ being a representative of $\mathcal{O}$. Let σ denote the spectrum of M. If $r > 0$ is given and there exists $\lambda \in \sigma$ with $r < |\lambda|$, let $C_{\leq r}$ and $C_{>r}$ denote the realified generalized eigenspaces of M associated with the nonempty disjoint spectral sets $\{\lambda \in \sigma; |\lambda| \leq r\}$ and $\{\lambda \in \sigma; r < |\lambda|\}$, respectively.

Theorem 4.2. (i) There exist a Floquet multiplier $\lambda_u > 1$ and $\psi_u \in \overset{\circ}{K} = \{\varphi \in C(\mathbb{K}); \varphi(\theta) > 0 \text{ for } \theta \in \mathbb{K}\}$ with $M\psi_u = \lambda_u \psi_u$.
(ii) $\dim C_{>1} = 1$.
(iii) $-1 \notin \sigma$.
(iv) For every $\lambda \in \sigma \setminus \{0, 1, \lambda_u\}$, $|\lambda| < 1$.

Theorem 4.2 (i) implies that the periodic orbit $\mathcal{O}$ is linearly unstable. Choose $\lambda \in (0, 1)$ so that

$$\lambda > \max\Big\{\frac{1}{\lambda_u}, \max_{\zeta \in \sigma, |\zeta| < 1} |\zeta|\Big\}.$$

Then there are convex open neighborhoods $N_{>1}$ of 0 in $C_{>1}$ and $N_{\leq 1}$ of 0 in $C_{\leq 1}$ and a C^1-map $w^u : N_{>1} \to C_{\leq 1}$ so that $w^u(0) = 0$, $Dw^u(0) = 0$ and $w^u(N_{>1}) \subseteq N_{\leq 1}$ and with $N^u = N_{>1} + N_{\leq 1}$, the shifted graph

$$W^u(p_0, \Phi(\omega, \cdot), N^u) = \{p_0 + \chi + w^u(\chi); \chi \in N_{>1}\}$$

coincides with the set of $\chi \in p_0 + N^u$ such that there is a trajectory $\{\chi^n\}_{-\infty}^0$ with $\chi^0 = \chi$, $\lambda^n(\chi^n - p_0) \in N^u$ for all $n \in -\mathbb{N}$ and $\lambda^n(\chi^n - p_0) \to 0$ as $n \to -\infty$. The unstable set $W^u(\mathcal{O})$ of the periodic orbit $\mathcal{O}$ is defined as

$$W^u(\mathcal{O}) = \bigcup_{t \geq 0} \Phi(t, W^u(p_0, \Phi(\omega, \cdot), N^u)).$$

Theorem 4.3. $W^u(\mathcal{O}) = bd\, W \setminus \{\xi_-, \xi_+\}$.

In summary, we showed that $\overline{W}$ contains a nontrivial periodic orbit $\mathcal{O}$. This periodic orbit is linearly unstable and its basin of attraction within $\overline{W}$ is the disk $\overline{W} \cap S$ minus the trivial equilibrium. Other orbits in $\overline{W} \setminus S$ are heteroclinic orbits from either 0 or $\mathcal{O}$ to the two nontrivial equilibria ξ_- and ξ_+.

Finally, we remark that the discrete Lyapunov functional theory developed in the recent work of Mallet-Paret and Sell [7, 8] played important roles in our discussion. Our present work presents some evidence to support the statements in the above work "(that the work [7, 8] opening the door to a general inquiry into the structure of the attractor of the (cyclic) systems" and to support the expectation in the featured review of Smith [10] that "further progress on this interesting class of systems in the future" (due to the work of Mallet-Paret and Sell [7, 8]).

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Chaos in Neural and Gene Networks with Hard Switching

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Abstract

Although there are ways of demonstrating the existence of chaotic dynamics on certain sets in nonlinear systems of ODEs, it is often very difficult to prove that such sets are attracting. In the context of neural networks and networks that model gene regulation or chemical kinetics, the use of a hard switching nonlinearity or Heaviside step function response for each network unit has been shown to give considerable analytic leverage. These 'Glass networks', named after their originator, have been shown to have an interesting range of possible dynamics in 4 or more dimensions, and a number of attempts have been made to pin down chaotic dynamics in particular examples. However, the existence of a chaotic attractor has never been proven in any such network, despite demonstration of the existence of transverse homoclinic points and sets on which the dynamics are homeomorphic to shifts of finite type. The problem has been that the sets supporting chaotic dynamics need not be attracting, so that long, stable periodic solutions remained a possibility. Here we present a method of proof that the dynamics on an attractor in a particular network is aperiodic and we demonstrate its application in three examples. The method uses matrix theoretic techniques which are sufficient to prove non-existence of stable periodic orbits in many chaotic networks particularly in 4 dimensions, though they may not be sufficient for every such network.

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1. Introduction

There has been an interest for some time now in the possibility of chaotic dynamics in neural networks, though its rôle remains controversial (e.g., [1, 2]) and its presence in real biological neural networks even more so [3]. Whatever its potential function, the approach to identifying chaos in network models has typically been numerical integration and sometimes indirect tests such as for positive Lyapunov exponents in time series generated by the network equations (e.g., [4, 5]). Analytic proof of stable (attracting) chaotic dynamics in neural network equations has so far been rare and dependent on special constructions [6, 7, 8, 2, 9]. In this paper we describe a general approach to proving the existence of attractors with at least aperiodic dynamics in a broad class of networks, and demonstrate it with some examples. We do not address here the notorious problem of identifying chaos in data obtained from physiological neural systems [3].

Neural network models are highly nonlinear, principally due to the ubiquitous sigmoidal functions that occur at all scales of description, from electrochemical models of membrane activity to coarse-scale models of abstracted neural activity in large networks, but also due in some cases to quadratic or higher order product terms. These nonlinearities and the large number of dimensions make it difficult to show analytically that a network has any particular dynamical behaviour. The existence of a Lyapunov functional for additive (Cohen-Grossberg-Hopfield) networks with symetric connections is an exceptional case that allows a rather beautiful proof that all trajectories converge to a fixed point [10, 11, 12]. In other contexts, particularly in attempts to deal with questions about physiological networks, fixed points are not necessarily expected and oscillation is ubiquitous. Proof of the existence of stable limit cycles in neural networks is generally much more difficult than proof of the existence of stable fixed points. Proof of the existence of stable chaotic dynamics, or any kind of aperiodic dynamics, is considerably more difficult still. Existing work on these lines seems to rely on constructions involving known chaotic equations or systems. Networks have been built, for example, out of chaotic components such as discrete time quadratic maps (e.g., [8, 2]). Wang [7] proved the existence of stable chaotic behaviour in a family of simple discrete-time two-neuron networks, but again the construction amounted to choosing an excitatory and an inhibitory response function so that together they act as a unimodel one-dimensional map (like the logistic map) known to be chaotic, an idea also exploited by Priesol et al. [13]. Lapedes and Farber [6] raised the idea of training a network (in their case a multi-layer perceptron network with backpropogation) to predict the dynamics of a known chaotic equation such as the logistic map or the Mackey-Glass equation. Here again, although a network with stable chaotic dynamics is the result, data from a known chaotic equation must be used to train the network in the first place.

Despite the general intractability of highly nonlinear systems like neural networks, considerable analysis, particularly for limit cycles, has proven to be possible in networks in which all interactions take the form of the 'high gain' limit of sigmoidal interaction — a step function [14, 15, 16]. Networks with steep sigmoids display switching behaviour as variables cross their thresholds. The high-gain limit could be called 'hard switching'. There is good evidence that dynamical behaviour of networks with sufficiently steep sigmoid interactions are qualitatively the same as the networks obtained by taking the limiting step function interactions [17, 4, 5], and at least some analytic results along these lines [18]. The hard-switching limit puts us into the context of what have been called 'Glass networks' after their originator [21]. The current work presents a method by which the existence of attracting aperiodic dynamics can be proven analytically at least in some networks of this type.

Glass networks [16] are sytems of ordinary differential equations in which all interactions between variables are of the hard switching type. They have been studied both in the context of neural networks and of gene regulatory and other biochemical networks (and are to be contrasted with the discrete time Boolean networks whose theory has been developed by Kauffman [19]). The occurrence of complex dynamics in Glass networks has been discussed in a number of papers [4, 5, 20, 21, 16, 22, 23]. Very complicated periodic orbits have been shown to exist in Glass networks of as few as 4 interacting elements [22, 24] and evidence has been found for chaotic dynamics, discussed further in Section 3 below. Sets on which the dynamics are chaotic have been shown to exist, but it has not been shown that these sets are attracting. The possibility has always been open that stable long periodic solutions exist in these networks despite the apparent irregularity of numerically computed solutions. Here, we present a proof that this is not the case in three example networks, including the one studied by Mestl et al. [20]. This is the first complete proof of the existence of attracting aperiodic dynamics in any Glass network.

Although the approach is indirect, in that it focusses on proving the non-existence of stable periodic orbits (or stable fixed points) in some trapping (invariant) region, and thus allows only the conclusion that the attractor in this region must be aperiodic, when this result is combined with the previous work indicating the existence of sets supporting chaotic dynamics, it seems clear that the dynamics on the attractor is indeed chaotic, rather than having some other kind of aperiodicity. Quasiperiodicity is possible in these networks but is a special behaviour depending on non-interaction of units involved in periodic orbits with non-commensurable periods, which is not the situation in any of the examples studied here or in any of the above-cited references.

2. Background

Glass networks can be expressed in the form

$$\dot{y}_i = -y_i + F_i(\tilde{y}), \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)', \qquad i = 1, \ldots, n, \tag{1}$$

where

$$\tilde{y}_i = \begin{cases} 0 & \text{if } y_i < 0 \\ 1 & \text{if } y_i > 0 \end{cases}, \tag{2}$$

and the prime (′) denotes matrix transposition. Additive neural networks with hard switching fall into this class. In that case,

$$F_i(\tilde{y}) = W\tilde{y} - \tau, \tag{3}$$

where W is the connection matrix, and τ is the vector of thresholds. Other neural network frameworks are not additive and may for example have quadratic terms. As long as all such interaction terms are passed through the nonlinearity, and if we take the hard switching nonlinearity (Heaviside step function), then these other neural networks still clearly fit into the framework of Equations 1 and 2. Note that thresholds at non-zero values and outputs of the switching function other than 0 and 1, can be accommodated into the above framework by transformation of variables even if these values are non-uniform. Non-uniform decay rates also do not present a major problem and the theory for Glass networks can be extended to cover them. While Equation 2 does not define $\tilde{y}_i$ when $y_i = 0$, trajectories in almost all cases can be extended in an unambiguous way to their limiting value as $y_i \to 0$ from either side. The only exceptions occur when a trajectory passes through a point where two or more variables are zero simultaneously in which case there may or may not be an unambiguous continuation, but such trajectories form a set of measure zero in $\mathbb{R}^n$ and we do not consider them further.

Solutions to these equations consist of continuous, piecewise-linear trajectories with corners where orthant boundaries are crossed in phase space. Within an orthant specified by sign structure $\tilde{y}$, a trajectory is a straight line directed towards a focal point, $f = F(\tilde{y})$. If the focal point for a particular orthant lies within that orthant, then it is a fixed point of the dynamics and any trajectory entering this orthant approaches the fixed point asymptotically. If there is no self-input in the network, i.e., if for all i, F_i does not depend on y_i, then the only other possibility is that a trajectory consists of a continuing sequence of piecewise-linear segments, which can be specified by the crossing points on the orthant boundaries. Throughout this paper we will assume that there is no self-input in our networks, though allowing self-input does not invalidate our results as long as our trajectories stay away from orthant boundaries that cannot be crossed ('black' and 'white' walls in the terminology used in [20]). We will also only be interested in trajectories that do not enter an orthant containing its own focal point, in which case no more crossings occur.

Thus, trajectories for an n-dimensional network (n-net) may be specified by a discrete mapping on the $(n-1)$-dimensional boundaries. These are fractional linear mappings of the form

$$y^{(k+1)} = M^{(k)} y^{(k)} = \frac{B^{(k)} y^{(k)}}{1 + \langle \psi^{(k)}, y^{(k)} \rangle}, \tag{4}$$

where

$$B^{(k)} = I - \frac{f^{(k)} e_j'}{f_j^{(k)}}, \qquad \psi^{(k)} = \frac{-e_j}{f_j^{(k)}}, \tag{5}$$

j is the index of the variable that switches on exiting the kth orthant along the trajectory, e_j denotes the standard basis vector in $\mathbb{R}^n$, and the angle brackets denote the Euclidean inner product ($\langle \psi, y \rangle = \psi' y$). This mapping can be reduced to $n-1$ dimensions by removing the ith column and jth row of the matrix $B^{(k)}$ and the ith element of all the vectors, where i is the index of the variable that switched on entry to the orthant.

Fractional linear mappings of this form are closed under composition. Thus, if a trajectory cycles from a given orthant boundary back to the same boundary, that cycle corresponds to a mapping

$$Mx = \frac{Ax}{1 + \langle \phi, x \rangle}, \tag{6}$$

where $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\phi \in \mathbb{R}^{n-1}$. Here and throughout this paper we use

$$x = y|_{(i)},$$

the vector y reduced to dimension $n-1$ by removing the ith variable, which is zero on the boundary on which the cycle is defined. We denote this starting boundary for a cycle $\mathcal{O}^{(0)}$.

Whenever it is possible to exit from an orthant by multiple boundaries (i.e. more than one variable may reach zero first) only part of a given entry boundary to the orthant will map to each possible exit boundary. Extrapolating, only part of a boundary will contain trajectories that follow a given sequence of orthants and return to that boundary, in general. This domain of definition for a cycle map may in fact be empty, in which case no trajectories follow that cycle, or may be the entire starting boundary, $\mathcal{O}^{(0)}$. In general, it is a subset of $\mathcal{O}^{(0)}$, defined by a set of linear inequalities, and we call it the returning cone, C, for the map. It can be calculated as follows:

$$C = \{y \in \mathcal{O}^{(0)} | Ry \geq 0\}, \tag{7}$$

where R is a matrix with one row for each alternate exit variable around the cycle, being the row vector

$$R_{i,\cdot} = -\frac{e_i'}{f_i^{(k)}} B^{(k)} B^{(k-1)} \ldots B^{(0)} \tag{8}$$

in each case.

As we will make considerable use of cones, we briefly give a few definitions here [25, chapter 1]. For a set $S \subseteq \mathbb{R}^n$, *the set generated by* S is the set of finite nonnegative linear combinations of elements of S:

$$S^G = \left\{ \sum_{i=1}^{m} c_i x_i, \quad \text{for some } c_i \geq 0, x_i \in S, m \quad \text{finite} \right\}.$$

A set K is a *cone* if $K = K^G$, so for any S, S^G is clearly a cone. Note that this definition implies that all cones are convex. Sometimes, the above is taken to define a *convex cone*, while a *cone* is defined simply as a set K for which $cK \subseteq K, \forall c \geq 0$ [26]. In any case, we are concerned mainly with convex cones, so the distinction matters little in our context. K is a proper cone if it is convex, closed, solid ($\text{int}(k) \neq \emptyset$) and pointed ($K \cap (-K) = \{0\}$). *Extremal* vectors of a proper cone are those required (up to nonnegative multiples) to generate it. A *polyhedral cone* is one that can be generated by a finite set S. For example, the returning cone, C, above is polyhedral, generated by intersections of the $(n-1)$-dimensional hyperplanes in the rows of $Ry = 0$ and the boundaries of $\mathcal{O}^{(0)}$.

Now we state without proof results on the existence and stability of periodic orbits in Glass networks [16].

Proposition 1. Trajectories starting on the same ray through the origin remain on the same ray through the origin under the discrete mapping. Furthermore, trajectories from points on the same ray converge under the discrete mapping as $t \to \infty$.

Note that it is possible for infinite iteration of the mapping to occur in finite time, when some variables are spiralling in to zero. In this case, coordinates corresponding to other variables will not have had time to converge and the infinite sequence of iterations of the map does not describe the trajectory for all time. However, the variables that have reached zero should then remain zero until some other variable switches and a new sequence commences, unless the multiplicity of variables at zero makes continuation ambiguous. If continuation is possible, then, trajectories starting from points on the same ray continue to converge (see [16]).

Proposition 2. The fractional linear map of Equation 6 maps straight lines to straight lines, and in general linear subspaces to linear subspaces.

Proposition 3. If v is an eigenvector of A in Equation 6, corresponding to eigenvalue $\lambda \neq 0$, such that $\langle \phi, v \rangle \neq 0$, then

$$y^* = \frac{(\lambda - 1)v}{\langle \phi, v \rangle} \tag{9}$$

is a fixed point of M and there is no other non-zero fixed point in the span of v.

Proposition 4. If the map M for a cycle has a fixed point in C, then it lies on an eigenvector of A with eigenvalue $\lambda > 1$. Conversely, if A has an eigenvector in C with eigenvalue $\lambda > 1$, then M has a fixed point in C.

Proposition 5. A fixed point y_i^* of the discrete map (Equation 6) is asymptotically stable if the corresponding eigenvalue λ_i of the matrix A is the unique dominant one ($\lambda_i > |\lambda_j|, j \neq i$); it is neutrally stable if λ_i is dominant but $\lambda_i = |\lambda_j|$ for some $j \neq i$; and it is unstable otherwise.

Note that for any periodic orbit, $\langle \phi, v \rangle > 0$ [16], because the denominator of the mapping (Equation 6) is the exponential of the period of the orbit. It may occur that $\langle \phi, v \rangle = 0$ for a fixed point, v, of the mapping. In that case, the "period" of the limiting orbit is 0, so that v is actually a stable fixed point. It must then be 0 in all variables that switch in the cycle. Trajectories spiral in to the fixed point. Note that in this case, $Av = v$, which implies that v is an eigenvector corresponding to eigenvalue 1.

On a given orthant boundary, $\mathcal{O}^{(0)}$, several cycles that return to the same boundary may be possible, each with its own returning cone. For example, we may have $M_A : C_A \to \mathcal{O}^{(0)}$ and $M_B : C_B \to \mathcal{O}^{(0)}$. The image of C_A under M_A may intersect with C_B and *vice versa*. Thus, trajectories may follow a complex sequence of cycles corresponding to repeated composition of M_A and M_B in some order. An example of such a situation occurs in the network studied in [20, 23]. Numerical investigations of a number of Glass networks of dimension 4 or more have appeared to show very complex sequences of this type. There may also be more than two mappings defined on a boundary as in the example studied in [16], and in Example 3 below.

3. The Question of Aperiodic Dynamics on Attractors

The above-mentioned examples appear to exhibit chaotic dynamics. Mestl et al. [20] were able to show that their example network had a trapping region, i.e., a cone on an orthant boundary that was invariant under the Poincaré map consisting of two regions, one on which M_A operated and one on which M_B operated. Each mapping had an unstable fixed point in its part of the trapping region. They showed that the stable and unstable manifolds of these fixed points intersected to produce transverse homoclinic points. This essentially guarantees that there is a set on which the dynamics are chaotic. However, they were not able to show that this set was attracting, and the possibility was left open that there might still be a very long and complicated periodic orbit that was attracting. Edwards et al. [23] showed that the symbolic sequences of A's and B's corresponding to trajectories could not contain the sequence BB, although bifurcation as a parameter changed could alter the dynamics to allow this symbolic subsequence.

Another network has been shown to contain a 'golden mean map', with topology homeomorphic to a shift of finite type, similar to the Smale horseshoe, but this only applied on a set that was certainly not itself an attracting set, since the dynamics actually involved many other cycle mappings besides the two that made up the golden mean map [16].

Numerical investigations of example additive networks suggest that dynamics of complicated trajectories are ergodic in the sense that they converge to an invariant distribution of values of each variable. Lewis and Glass [4, 5] showed by numerical simulation that in these examples, the invariant density across many different initial conditions evolved over time to match the invariant density along a single trajectory. This suggests the existence of a nontrivial attractor with sensitive dependence on initial conditions. Lewis and Glass also calculated Lyapunov exponents for their integrated trajectories and found good evidence for a positive exponent, again suggestive of chaos. The difficulty with these methods is that they rely on numerical integrations and do not constitute a proof. There is no guarantee that the trajectories calculated are not on long transients and have not reached an attractor, that they are not on a very complex periodic orbit or that accumulation of numerical errors is not critical.

While it has seemed virtually certain for some time that these examples do exhibit stable chaotic dynamics, this has not been proven. While long stable periodic orbits seem unlikely in these examples, they have not been ruled out. It should be noted that it is possible, even in networks of only four elements, to have long and complex but stable limit cycles. Even in the class of networks in which the focal point coordinates F_i are all ± 1, 4-nets have been found with stable limit cycles consisting of up to 252 switchings (or linear trajectory segments) [22]. Gedeon has constructed 4-net examples in the general class (arbitrary F_i) with stable limit cycles consisting of arbitrarily many switchings [24]. These results make it unclear whether or not such long periodic attractors exist in the apparently chaotic networks. While negative divergence implies the contraction of phase space and the existence of attractors of dimension $< n$ [16], the question remains: Are attracting aperiodic dynamics possible in Glass networks (aside from obvious but special constructions of quasiperiodic attractors)?

4. Matrices and Invariant Cones

Proposition 1 implies that the radial component of a cycle map's dynamics can be factored out, unless trajectories converge to the origin. Except in the latter special case, we can ignore the radial component and consider the discrete map (Equation 6) to be essentially a matrix multiplication acting on a ray:

$$Mx = Ax, \tag{10}$$

where x may now be taken to represent an entire ray, rooted at the origin. The scalar division in the fractional linear map only moves a point in or out on the ray. The results outlined in the previous section imply that there is a stable periodic orbit following a given cycle if its matrix, A, has its dominant eigenvalue, λ_1, real and > 1 and the corresponding dominant eigenvector (which must then be real) in its returning cone, C. A stable spiral fixed point is a degenerate case of such a periodic orbit with $\lambda_1 = 1$. Complex cycles, made of compositions of a pair of maps, for example M_A and M_B, can now be considered as a 'word' made up of A's and B's (being in fact a matrix product, as well as a symbolic sequence). In order for such a word, W, to correspond to a stable periodic orbit, the dominant eigenvector of W must lie in its returning cone. This is a statement about the location of the dominant eigenvector of a matrix product. While there is no general result in linear algebra for the location of eigenvectors of a matrix product in terms of the eigenvectors of the individual matrices (they can go anywhere), there *are* results dealing with the location of a dominant eigenvector that we can use, the most obvious being the Perron-Frobenius theorem and generalizations.

A finite-dimensional version of the Krein-Rutman theorem generalizes the Perron-Frobenius theorem from nonnegative matrices to any matrix that leaves a proper cone invariant. We give two versions of this result (adapted from Berman and Plemmons [25, Chapter 1]).

Theorem 6. If $K \in \mathbb{R}^n$ is a proper cone and $AK \subseteq K$, where $A \in \mathbb{R}^{n\times n}$, then the spectral radius, $\rho(A)$, is an eigenvalue and a corresponding eigenvector lies in K.

Theorem 7. If $K \in \mathbb{R}^n$ is a proper cone and $A(K - \{0\}) \subseteq \text{int}K$ (i.e., A is K-positive) then $\rho(A)$ is a simple eigenvalue and its corresponding one-dimensional eigenspace lies in $\text{int}(K \cup (-K)) \cup \{0\}$.

It is easy to see that if $AK \subseteq K$ and $BK \subseteq K$, where K is a proper cone, then not only must eigenvectors corresponding to $\rho(A)$ and $\rho(B)$ lie in K but the same must be true of any matrix W made up of a product of A's and B's in any order, since each consecutive multiplication by A or B maps K into K.

Proposition 8. Let $(A^*B^*)^*$ be the set of all finite matrix products of A's and B's in any order. If $AK \subseteq K$ and $BK \subseteq K$, where K is a proper cone, then $WK \subseteq K$, $\rho(W)$ is an eigenvalue of W, and a corresponding eigenvector lies in K, for any $W \in (A^*B^*)^*$.

We wish to argue that no eigenvector corresponding to $\rho(W)$ can lie anywhere other than $K \cup (-K)$. The above result is sufficient as long as $\rho(W)$ is a simple eigenvalue of W. Otherwise, $\rho(W)$ may have an eigenspace of dimension > 1 and the existence of an eigenvector in K is not sufficient to preclude an eigenvector elsewhere. Even if $\rho(A)$ and $\rho(B)$ are simple eigenvalues of A and B, it is not immediate that this is true for every product W. For any W for which $W(K - \{0\}) \subseteq \text{int}(K)$, however, we can apply Theorem 7.

Proposition 9. Suppose a union of (solid but not necessarily proper) cones, $T = T_A \cup T_B$, of an orthant boundary, $\mathcal{O}^{(0)}$, is invariant under the network dynamics consisting of two cycle maps, $M_A : T_A \to T$ and $M_B : T_B \to T$. Suppose there exists a proper cone K such that $AK \subseteq K$ and $BK \subseteq K$, and that $(K \cup (-K)) \cap T = \{0\}$. Then there is no stable periodic orbit through T for any cycle $W \in (A^*B^*)^*$ for which $W(K - \{0\}) \subseteq \text{int}(K)$.

Proof. Under the network dynamics, every point $x \in T$ returns to T repeatedly after following either cycle A or B each time. T is a Poincaré section for the Poincaré map,

$$Mx = \begin{cases} M_A x & \text{if } x \in T_A \\ M_B x & \text{if } x \in T_B \end{cases} . \tag{11}$$

Thus, each point (or ray) in T has a corresponding symbolic sequence in $(A^*B^*)^*$. Suppose now that there is a stable periodic orbit through T. Then this corresponds to a non-zero fixed point $x \in T$ of a composite mapping whose matrix is some $W \in (A^*B^*)^*$. By Propositions 4 and 5, x is a dominant eigenvector of W, and its corresponding eigenvalue is $\rho(W) > 1$. However, since K is a common invariant proper cone for both A and B, any $W \in (A^*B^*)^*$ has a dominant eigenvector in K (and $-K$), by Proposition 8. Furthermore, if $W(K - \{0\}) \subseteq \text{int}(K)$, then this dominant eigenvector is unique up to scalar multiples, by Theorem 7. This contradicts the assumption that $(K \cup (-K)) \cap T = \{0\}$, i.e., that the only point in the intersection of T and $K \cup (-K)$ is the origin. x cannot be in both T and $K \cup (-K)$.

Remark 1. The conditions of the theorem are not contradictory. We are supposing that K is invariant for A and B individually. We do not assume that T is invariant for A or B, rather that it is invariant for M which is defined so that A applies on T_A and B applies on T_B. It is entirely possible for this arrangement to occur — see the examples in Section 5

Remark 2. Note that if $-A$ and B leave a proper cone, K, invariant or map it into its interior then the above theorems apply to $((-A)^*B^*)^*$, and since any $W \in (A^*B^*)^*$ is ± an element of $((-A)^*B^*)^*$, we have that $\pm\rho(W)$ is an eigenvalue of W and the corresponding eigenvector still lies in $K \cup (-K)$. The same is clearly true if $-B$, rather than B, leaves K invariant.

Remark 3. It is easy to see that a similar result also holds when there are more than two cycles involved, with $T = \bigcup_i T_i$ and $M_i : T_i \to T$ if K is invariant simultaneously for all A_i (the matrix of M_i).

Remark 4. In practice, this result implies that words W that do not map K into its interior, i.e., that map non-zero boundary points to non-zero boundary points of K, have to be checked separately. In each case if $\rho(W)$ is simple or if it is not simple but still no corresponding eigenvector lies in T then there is no stable periodic orbit corresponding to the cycle W.

If trajectories are trapped in a trapping region, T, and no stable limit cycles pass through T then the only remaining possibilities for the dynamics are convergence to a fixed point or aperiodicity. Convergence to a fixed point other than the origin implies a spiral point on the boundary of $\mathcal{O}^{(0)}$, which is covered as a degenerate case under the results on stable periodic orbits, so the only further check required is for convergence to the origin. (Recall that we have ruled out trajectories ending at a fixed point in the interior of an orthant). There is always an attractor of dimension $< n$, since the flow of these networks (under the condition of no self input) is contracting. From the point of view of the Poincaré map, the determinant of the matrix A in the map is 1 (under the no self input condition) and the denominator of the map is always greater than 1, so volumes contract [16].

The question of convergence to the origin is dealt with by the following proposition.

Proposition 10. Let $T = T_A \cup T_B$ be a trapping region on an orthant boundary for a network such that cycle maps M_A and M_B are defined on polyhedral cones T_A and T_B respectively. Let A be the matrix in the map M_A (Equation 6) and B be the matrix in M_B. Let $x_i, i = 1, \ldots, k$ and $y_i, i = 1, \ldots, m$ be the extremal vectors of cones T_A and T_B respectively with unit L^1 norm. If $||Ax_i||_1 > 1$ for $i = 1, \ldots, k$ and $||By_i||_1 > 1$ for $i = 1, \ldots, m$, then no trajectory in T can converge to the origin.

Proof. Consider one of the unit extremal vectors x_i. Let

$$0 < c < \frac{||Ax_i||_1 - 1}{\phi' x_i} = c_i^*.$$

Then

$$||M_A(cx_i)||_1 = \frac{||A(cx_i)||_1}{|1 + \phi'(cx_i)|} = \frac{c||Ax_i||_1}{1 + c\phi' x_i}$$

since $1 + \phi'(cx_i) > 1$ when $x_i \neq \mathbf{0}$ (see note following Proposition 5). Our assumed range for c implies that

$$1 + c\phi' x_i < ||Ax_i||_1,$$

so

$$||M_A(cx_i)||_1 > c = ||cx_i||_1.$$

Thus, the norm of cx_i increases under the action of M_A for small positive c. The same is true for each i. If we let $c^* = \min_{i=1,\ldots,k} c_i^*$ then the norm of cx_i increases under the action of M_A for all i if $0 < c < c^*$. Now consider any point, z, in T_A such that $||z||_1 = c$ with $0 < c < c^*$. The set of points, $T_{A,c}$, in T_A whose L^1 norm is c clearly lie on a hyperplane. The points cx_i lie in this hyperplane and we know that their norm is increased under M_A. Thus, using Proposition 2 the image under M_A of $T_{A,c}$ lies in a hyperplane further from the origin. That is, every point in the image has norm $> c$. An exactly parallel argument can be made for the extremals of T_B under the action of

M_B, so there is a positive constant d^* such that for any point $z \in T_B$ for which $0 < ||z||_1 < d^*$ we have $||M_B z||_1 > ||z||_1$. Clearly no convergence to the origin is possible for any trajectory in T, since whenever the norm of a point on the trajectory goes below $\min\{c^*, d^*\}$ it must increase on the next iteration.

Again, extension to more than two cycles is straightforward.

Then, finally, as a corollary to Propositions 9 and 10 we have the following:

Corollary 11. Under the conditions of Propositions 9 and 10, if for every $W \in (A^*B^*)^*$, one of the following holds:

(a) $W(K - \{0\}) \subseteq \text{int}(K)$,

(b) $\rho(W)$ is simple, or

(c) no eigenvector corresponding to $\rho(W)$ lies in T,

then the network has an attractor passing through T, on which the dynamics must be aperiodic.

5. Examples

The results of the previous section allow us to tackle the problem of proving existence of stable aperiodic dynamics in a network by finding a trapping region on an orthant boundary (Poincaré section) involving two or more cycles, showing that trajectories in this region do not converge to the origin, and showing that the matrices for these cycles have a common invariant proper cone that does not intersect the trapping region (though they may share a common boundary, if we can be sure that the dominant eigenvector never lies on this shared boundary). How to find a common invariant proper cone even for a pair of matrices is a non-trivial problem that does not seem to have been addressed in the linear algebra literature. Some results for this question have been obtained at least under some conditions (particularly in $\mathbb{R}^3$) and will appear in a future publication. Here, we will simply demonstrate the existence of common invariant cones in a few examples in $\mathbb{R}^3$, but these should make the essential ideas of the method clear.

The basic idea (for matrices in $\mathbb{R}^3$, at least) is to look at intersections of invariant linear subspaces spanned by subsets of eigenvectors for each matrix, or in some cases intersections with the images of these subspaces under the action of the other matrix. For a matrix with positive real distinct eigenvalues, the cone generated by its eigenvectors, for example $\{u, v, w\}^G$ in $\mathbb{R}^3$, is clearly invariant, no matter which of the 2^n choices of signs of the eigenvectors one picks. Such cones form a partition of $\mathbb{R}^n$ similar to that of the 2^n standard orthants of $\mathbb{R}^n$. If one of the non-dominant eigenvalues is negative, then we need to put both positive and negative multiples of its eigenvector in the cone's generating set to make it invariant. This cone then comprises two of the original 2^n cones. It is also no longer a proper cone, since it is not pointed (i.e., $K \cup (-K) \neq \{0\}$), so we will not be able to use the whole of it as a common invariant cone. If other eigenvalues are negative, they too will have a pair of eigenvectors of opposite sign in the generating set. If a common invariant proper cone exists for a pair of matrices, the easiest kind of candidate to check is the intersection of cones generated by the eigenvectors of each matrix. The extremals (i.e., the generating set) for this cone of intersection, will in general be the intersections of invariant subspaces of the two matrices. Our first example, below, is of this type. Such an intersection cone will not always be invariant for both matrices, however, and other candidates may have to be found, using images of invariant subspaces of one matrix under multiplication by the other matrix, or images of other extremals found by intersection. Our second example is a relatively simple one of this type. Our third example, an additive neural network, is more complicated and requires finding a common invariant proper cone for seven matrices simultaneously. This task is made somewhat easier by the fact that the matrices fall into groups with very similar eigenvalues.

5.1. Example 1

The 4-net considered by Mestl et al. [20] is given by the following set of differential equations

$$\begin{aligned}
\dot{y}_1 &= -y_1 + 2(\tilde{y}_2\bar{\tilde{y}}_3 + \bar{\tilde{y}}_2\tilde{y}_3) - 1.2546 \\
\dot{y}_2 &= -y_2 + 2(\bar{\tilde{y}}_1\bar{\tilde{y}}_4 + \tilde{y}_1\tilde{y}_4) - 1.3762 \\
\dot{y}_3 &= -y_3 + 2\tilde{y}_1\tilde{y}_2 - 0.8024 \\
\dot{y}_4 &= -y_4 + 2(\bar{\tilde{y}}_1\bar{\tilde{y}}_3 + \tilde{y}_3) - 1.2682
\end{aligned} \tag{12}$$

where $\bar{\tilde{y}}_i = 1 - \tilde{y}_i$ and $\tilde{y}_i$ is as defined in Equation 2.

The focal point coordinates, $F_i(\tilde{y})$, for a given orthant $\tilde{y}$ may be calculated from the equations. For example, when y is in the positive orthant, we have $\tilde{y} = (1, 1, 1, 1)$ and $F = (-1.2546, 0.6238, 1.1976, 0.7318)$.

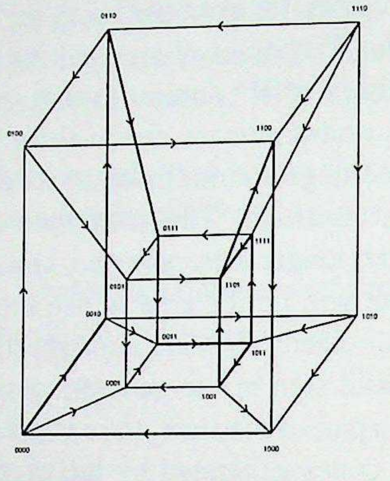

Figure 1. The logical structure of the network from Mestl et al. [20]. Vertices represent orthants of phase space and edges represent the direction of flow across boundaries between orthants.

The logical structure of the network, i.e., the directions of flow across orthant boundaries, can be depicted as a directed graph on a four-dimensional hypercube, or 4-cube (Figure 1). The vertices represent orthants of phase space and the directed edges represent the direction of flow across the boundary between a pair of orthants. The vertex labels correspond to the value of $\tilde{y}$ in each orthant (essentially the sign structure of the orthant).

Starting on the edge between 0011 and 1011, we encounter the following vertices in defining two cycles:

$$\begin{aligned} A &: 1011 \to 1111 \to 1101 \to 1100 \to 1000 \to 0000 \to 0010 \to 0110 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \to 0111 \to 0011, \\ B &: 1011 \to 1001 \to 1101 \to 1100 \to 1000 \to 0000 \to 0010 \to 0110 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \to 0111 \to 0011. \end{aligned}$$

The labels A and B can be taken to be the matrices in the two mappings corresponding to these two cycles. Letting $x = (x_1, x_2, x_3)' = (y_2, y_3, y_4)'$, the mappings are

$$M_A x = \frac{Ax}{1 + \langle \phi, x \rangle}, \quad M_B x = \frac{Bx}{1 + \langle \psi, x \rangle}, \tag{13}$$

where

$$\begin{aligned}
A &= \begin{pmatrix} -10.613261 & -9.804502 & -3.452968 \\ 2.105432 & 4.747100 & 2.991261 \\ 6.394689 & 5.771168 & 1.934729 \end{pmatrix}, \\
\phi &= (4.919877, 11.494663, 6.269858)', \\
B &= \begin{pmatrix} 0.400545 & -1.242174 & -3.452968 \\ 4.743815 & 6.798225 & 2.991261 \\ -0.264146 & 0.594472 & 1.934729 \end{pmatrix}, \\
\psi &= (9.789031, 15.280029, 6.269858)'.
\end{aligned}$$

The details of how these mappings are calculated can be found in [20] or [16] but note that there is a typographical error in Equation (4.3) in [20]. Since we start on the boundary between the 0011 and 1011 orthants, the maps apply to the boundary $(0, -, +, +)$ in $\mathbb{R}^4$ or the $(-, +, +)$ octant of $\mathbb{R}^3$ and are only defined on a cone in this octant, their respective returning cones. The eigenvalues and corresponding eigenvectors of matrix A are

$$\begin{aligned}
\lambda_1 &= -8.291439, \qquad \lambda_2 = 4.387496, \qquad \lambda_3 = -0.027489, \\
u_A &= \begin{pmatrix} -11.134116 \\ 0.230442 \\ 6.832402 \end{pmatrix}, v_A = \begin{pmatrix} -0.244180 \\ 0.326903 \\ 0.132569 \end{pmatrix}, w_A = \begin{pmatrix} 0.443301 \\ -0.842145 \\ 1.032191 \end{pmatrix},
\end{aligned}$$

and similarly for matrix B

$$\begin{aligned}
\mu_1 &= 5.652616, \qquad \mu_2 = 3.429296, \qquad \mu_3 = 0.051588, \\
u_B &= \begin{pmatrix} -0.128914 \\ 0.359718 \\ 0.066676 \end{pmatrix}, v_B = \begin{pmatrix} -0.313318 \\ 0.289705 \\ 0.170607 \end{pmatrix}, w_B = \begin{pmatrix} 0.812294 \\ -0.722844 \\ 0.342127 \end{pmatrix}.
\end{aligned}$$

When the eigenvalues are distinct we use the convention that u, v and w correspond to the eigenvalues in order of decreasing modulus, and likewise the indices of the eigenvalues in the order $1, 2, 3$. The eigenvectors above are in fact fixed points of the respective maps M_A and M_B, the unique ones on each eigenvector, from Equation 9. Note that only u_A, v_A, u_B and v_B are in the orthant boundary to which the map applies.

Using Equations 7 and 8 the returning cone, C_A, for the A matrix is defined by

$$R_A = \begin{pmatrix} -8.886421 & -4.717095 & -0.016935 \\ 2.264171 & 1.680946 & 0.297248 \\ 1.603078 & 1.246261 & 0.000000 \end{pmatrix},$$

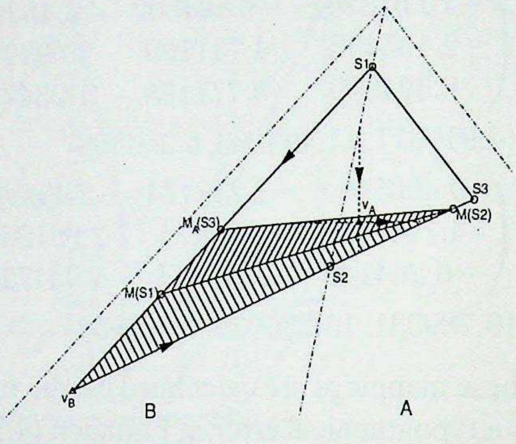

Figure 2. Sketch of the trapping region in the $(0, -, +, +)$ boundary for the network of Equation 12, projected onto a plane orthogonal to vector v_A, the second eigenvector of matrix A in the Poincaré mapping for the A cycle. Based on Fig.4(b) in Mestl et al. [20]).

and C_B is defined by

$$R_B = \begin{pmatrix} 6.346434 & 7.125196 & -0.016935 \\ -0.154213 & -0.199149 & 0.297248 \\ -1.603078 & -1.246261 & 0.000000 \end{pmatrix}.$$

Thus, for example, $x \in C_A$ if $R_A x \geq \mathbf{0}$ (i.e., is nonnegative).

A calculation now shows that $u_A \notin C_A$ and $u_B \notin C_B$, but $v_A \in C_A$ and $v_B \in C_B$. Therefore, v_B is an unstable fixed point of the B cycle, since it corresponds to an eigenvalue of B that is not the dominant one (Proposition 5). Similarly, v_A is an unstable fixed point of the A cycle.

This network has a trapping region in the $(0, -, +, +)$ boundary [20], shown schematically in Figure 2. The point v_B, called F_{B3} in [20], (note that the labels F_{B1} and F_{B3} are transposed in Table 2 in [20]), small discrepancies in some of the numbers given here and in [20] are most likely due to their reporting only four decimal places in the constants in their differential equations, when they may have actually used more in some of their calculations. We stick here to the values exactly as written in their equations, and v_A, called F_{A3} in [20], are shown in the figure. The stable and unstable manifolds of these points are indicated by the arrows. In the figure, everything is projected onto a plane orthogonal to the vector v_A, so the planar regions really represent cross sections of cones and points in the figure represent rays emanating from the origin. Some license has been taken in distorting the figure to make the regions clearer—the shaded region is actually very thin. It can be shown that the fractional linear maps associated with trajectories in these networks take straight lines to straight lines (Proposition 2), and this applies also to projections onto a plane [16]. The line through S_1 and S_2 is the separating boundary between the domains of definition (i.e., returning cones) of the A and B cycles (actually a plane projected onto a line in the figure). The dot-dash lines in Figure 2 indicate the domains of definition of these two cycles. The large letters show which cycle applies in which region. S_1 is the intersection of the stable manifold, W_B^s, of v_B and the separating boundary (again projected onto the plane). S_2 is the intersection of the unstable manifold, W_B^u, of v_B and the separating boundary. S_3 is the intersection of W_B^u and the inverse image under M_A of W_B^s. Let T_A be the triangular region (the projection of a triangular cone) with vertices S_1, S_2 and S_3, let T_B be the triangular region with vertices v_B, S_1 and S_2, and let $T = T_A \cup T_B$. Then T is mapped into the shaded regions. Note that by their definitions, $M_B v_B = v_B$, $M_B S_1$ and $M_A S_3$ lie on W_B^s, and $M_B S_2$ lies on W_B^u. Also, it can be checked that M_A and M_B coincide on the boundary between their domains. Thus, T is invariant under the network dynamics.

This establishes the existence of the trapping region, as described by Mestl et al. [20]. We now show that the matrices A and B have a common invariant proper cone. Again, we use a projection in order to visualize the regions under consideration, but for this purpose we find it more convenient to project onto the unit sphere because more than one octant will typically be involved. Thus, polyhedral cones in $\mathbb{R}^3$ become polygonal regions on the surface of the unit sphere.

Recall the notation for a cone as generated by a set of vectors, so that for example

$$\{u, v\}^G = \{x | x = c_1 u + c_2 v \text{ for some } c_1 \geq 0, c_2 \geq 0\}.$$

Let

$$\begin{aligned} z_1 &= \{-u_B, w_B\}^G \cap \{u_A, -v_A\}^G, \\ z_2 &= \{v_B, w_B\}^G \cap \{u_A, v_A\}^G. \end{aligned}$$

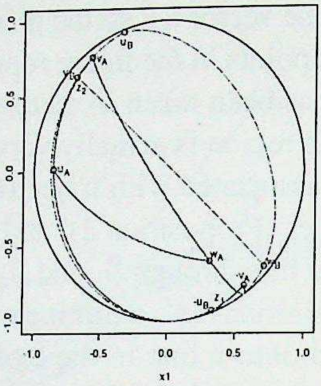

Figure 3. Invariant cone for the matrices A and B corresponding to Poincaré maps for the network of Equation 12, projected onto the unit sphere. Projections of the invariant subspaces through eigenvectors of A are indicated by solid lines. Projections of the invariant subspaces through eigenvectors of B are indicated by dashed lines.

Then we claim that the cone generated by v_B, $-u_B$, z_1 and z_2, i.e.,

$$K = \{v_B, -u_B, z_1, z_2\}^G,$$

is invariant for both $-A$ and B (see Figure 3). Since matrix multiplication is linear, to show that the cone K is invariant, we need only check that its extremal vertices are mapped back into the cone under each matrix multiplication. Clearly, $Bv_B = \mu_2 v_B$ so the ray through v_B is mapped to itself under multiplication by B. Similarly, $B(-u_B) = -\mu_1 u_B$. Since $\{v_B, w_B\}^G \in W_B^s$ and $\mu_2 > 0$, we have $Bz_2 \in \{v_B, z_2\}^G$. Similarly, since $\{-u_B, w_B\}^G$ is part of the stable invariant manifold of $-u_B$ and $\mu_1 > 0$, we have $Bz_1 \in \{-u_B, z_1\}^G$. Thus, $BK \subseteq K$.

Now, multiplying by $-A$, it is clear that $-Az_1 \in \text{span}(u_A, v_A)$ and a calculation shows that in fact $-Az_1 \in \{u_A, z_2\}^G$ (note that $-A$ has eigenvalue $-\lambda_2 < 0$ so there is reflection about $\{u_A, w_A\}^G$). Similarly, $-Az_2 \in \text{span}(u_A, v_A)$ and a calculation shows that $-Az_2 \in \{u_A, z_1\}^G$. Since multiplication by $-A$ contracts towards $\text{span}(u_A, v_A)$, it is clear that the points v_B and $-u_B$ will be taken closer to $\text{span}(u_A, v_A)$. A calculation is needed again, however, to verify that they are taken inside the narrow sides of the cone. This then implies that $-AK \subseteq K$, and therefore $AK \subseteq (-K)$. These points are shown in Figure 3.

Furthermore, while B takes every extremal point of K to the boundary of K and A takes two of the extremal points to the boundary of K, it is not difficult to see that if a cycle involves both A and B, all extremal points will be taken to $\text{int}(K)$. The points Az_1 and Az_2 are taken by B to the interior of K and v_B and $-u_B$ are already taken to $\text{int}(K)$ by A. We have already checked that A and B themselves do not have a dominant eigenvector in T_A or T_B respectively, and thus do not correspond to stable periodic orbits, so that only leaves us to check combinations, W, involving both A and B. These satisfy the condition $W(K - \{\mathbf{0}\}) \subseteq \text{int}(K)$.

In order to apply Proposition 9, we need to show that $(K \cup (-K)) \cap T = \{\mathbf{0}\}$. This is not quite true here, since $K \cap T = \{v_B, z_2\}^G$. However, there is actually a smaller trapping region within T that does not include this boundary [23]. Also, any point on this boundary is mapped under $-A$ to the interior of the invariant cone, and no point from the interior of the cone can be mapped back to this boundary under either $-A$ or B. Therefore, points on this boundary could not be eigenvectors for multiple products of matrices $-A$ and B, except in the case that the product consists only of the B matrix, in which case the eigenvector on the boundary is v_B itself, which we already know is not dominant.

Either way, we have established that there exist a disjoint T and $K \cup (-K)$. By Proposition 9, there cannot exist a stable periodic orbit through T. Thus, the dynamics on the attractor in this trapping region for the matrices acting on rays, or equivalently for the attractor's projection onto the plane of Figure 2, are aperiodic. However, we must still show that trajectories do not converge to the origin. This can be confirmed by checking that $||Ax_i|| > 1$ and $||By_i|| > 1$ for unit vectors x_i and y_i at the corners of the regions T_A and T_B, respectively. The unit extremal vectors can be calculated by intersections of the planes defined by rows of R_A and R_B with each other and with the boundaries of the octant. For example, v_B is an extremal vector of the region T_B, and

$$B\frac{v_B}{||v_B||_1} = 3.429435 > 1.$$

The other calculations are similar and show by Proposition 10 that trajectories in T cannot converge to the origin. Finally, by Corollary 11, trajectories through T converge to an attractor on which the dynamics are aperiodic.

5.2. Example 2

Another 4-net with apparently irregular dynamics is given by the differential equations

$$\begin{aligned}
\dot{y}_1 &= -y_1 + 2(\bar{\tilde{y}}_3\tilde{y}_4 + \tilde{y}_2\tilde{y}_3) - 1 \\
\dot{y}_2 &= -y_2 + 2(\tilde{y}_1\bar{\tilde{y}}_3\tilde{y}_4 + \bar{\tilde{y}}_1\tilde{y}_3\tilde{y}_4 + \bar{\tilde{y}}_1\bar{\tilde{y}}_3\bar{\tilde{y}}_4) - 1 \\
\dot{y}_3 &= -y_3 + 2(\bar{\tilde{y}}_1\tilde{y}_2 + \tilde{y}_1\tilde{y}_4) - 1 \\
\dot{y}_4 &= -y_4 + 2(\tilde{y}_2\bar{\tilde{y}}_3 + \bar{\tilde{y}}_1\tilde{y}_3) - 1
\end{aligned} \tag{14}$$

where again $\bar{\tilde{y}}_i = 1 - \tilde{y}_i$. Note that for this network the focal point coordinates are all ± 1. If we consider a Poincaré section on the boundary $(+,+,+,0)$, there are two cycles of interest, which we again call A and B:

$$\begin{aligned}
A &: 1110 \to 1010 \to 0010 \to 0000 \to 0100 \to 0110 \to 0111 \to 1111, \\
B &: 1110 \to 1010 \to 0010 \to 0011 \to 0001 \to 0000 \to 0100 \to 0101 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \to 0111 \to 1111,
\end{aligned}$$

with Poincaré mappings defined by Eqs. (13), where now

$$\begin{aligned}
x &= (x_1, x_2, x_3)' = (y_1, y_2, y_3)', \\
A &= \begin{pmatrix} -3 & -8 & 4 \\ -2 & -5 & 2 \\ -4 & -12 & 7 \end{pmatrix}, \\
\phi &= (-4, -14, 10)', \\
B &= \begin{pmatrix} 5 & 8 & 0 \\ 6 & 11 & -2 \\ 12 & 20 & -1 \end{pmatrix}, \\
\psi &= (12, 18, 2)'.
\end{aligned}$$

The eigenvalues of A are $\lambda_1 = -4.627213$, $\lambda_2 = 3.685846$ and $\lambda_3 = -0.058633$. The eigenvalues of B are $\mu_1 = 12.532871$, $\mu_2 = 2.434352$ and $\mu_3 = 0.032777$. The returning cones, C_A and C_B, for the A and B matrices respectively are determined by

$$R_A = \begin{pmatrix} -2 & -5 & 2 \\ 2 & 4 & -1 \end{pmatrix}, \quad R_B = \begin{pmatrix} 6 & 11 & -2 \\ -2 & -4 & 1 \end{pmatrix}.$$

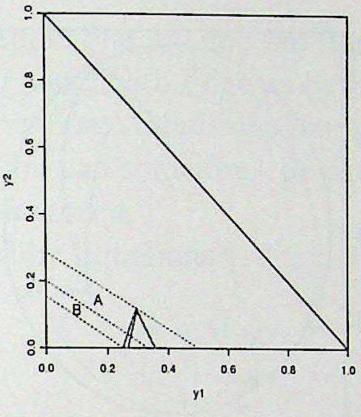

Figure 4. Trapping region in the $(+,+,+,0)$ boundary, for the network of Equation 14, projected onto a plane orthogonal to vector $(1,1,1)'$, with only the y_1 and y_2 components plotted. The returning cones for the A cycle and the B cycle are indicated by dotted lines and the images of these regions are indicated by solid lines. Note that the images lie withing the union of the returning cones.

These define cones:

$$\begin{aligned} C_A &= \left\{(0,\tfrac{2}{7},\tfrac{5}{7})',(\tfrac{1}{2},0,\tfrac{1}{2})',(\tfrac{1}{3},0,\tfrac{2}{3})',(0,\tfrac{1}{5},\tfrac{4}{5})'\right\}^G, & (15)\\ C_B &= \left\{(0,\tfrac{2}{13},\tfrac{11}{13})',(\tfrac{1}{4},0,\tfrac{3}{4})',(\tfrac{1}{3},0,\tfrac{2}{3})',(0,\tfrac{1}{5},\tfrac{4}{5})'\right\}^G, & (16)\end{aligned}$$

on which the Poincaré maps are defined for each cycle. We take $T_A = C_A$ and $T_B = C_B$. A has only its second eigenvector, v_A (corresponding to the eigenvalue with second largest modulus), in its returning cone, T_A. B has none in T_B. It turns out that $T = T_A \cup T_B$ is a trapping region for the network. This can be established simply by verifying that $M_A(T_A) \subseteq T$ and $M_B T_B \subseteq T$. The regions T_A and T_B as well as their images are depicted in Figure 4.

Once again, we use cones defined by eigenvectors of the A and B matrices and their images, to construct a common invariant proper cone, K (Figure 5). Again, we use $-A$, since $\lambda_1 < 0$. We define

$$\begin{aligned} z_1 &= \{u_B, v_B\}^G \cap \{u_A, w_A\}^G),\\ z_2 &= \{u_B, w_B\}^G \cap \{u_A, v_A\}^G. \end{aligned}$$

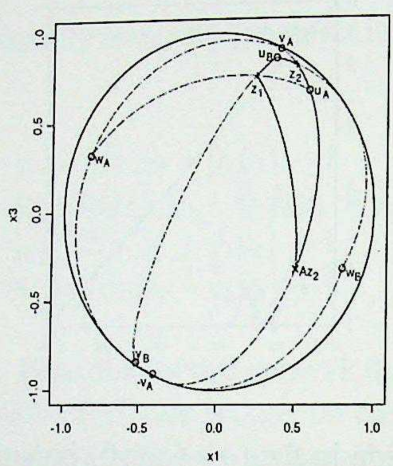

Figure 5. Invariant cone for the matrices A and B corresponding to Poincaré maps for the network of Equation 14, projected onto the unit sphere, indicated by solid lines. Projections of the invariant subspaces through eigenvectors are indicated by dashed lines.

Then we claim that

$$K = \{u_B, z_2, -Az_2, z_1\}^G$$

is an invariant cone for both A and B. Clearly, $Bu_B = \mu_1 u_B$, $Bz_1 \in \{u_B, z_1\}^G$ and $Bz_2 \in \{u_B, z_2\}^G$. A calculation shows that $B(-Az_2) \in \int(K)$. Also, $-Az_1 \in \int\{u_A, z_1\}^G \subseteq \int(K)$, since $-\lambda_1 > 0$ and $-\lambda_3 > 0$. By definition, $-Az_2 \in K$. While $-A(-Az_2) \in \text{span}(z_2, -Az_2)$, a calculation is needed to confirm that $-A(-Az_2) \in \{z_2, u_A\}^G$. Finally, another calculation shows that $-Au_B \in \int(K)$. See Figure 5.

Again, although A and B individually map some extremal points to the boundary of K, calculation of $-ABz_1$, $-ABz_2$ as well as $-Au_B$ mentioned above shows that all are in int(K), as is $B(-Az_2)$ so any cycle involving both A and B maps K to int(K). We have already noted that A and B individually do not have dominant eigenvectors in T.

Thus, $K \cup (-K)$ is invariant for both A and B, and is disjoint from T. Proposition 9 then guarantees that there are no stable periodic orbits through the trapping region T. We omit the straightforward calculations showing that $||Ax_i||_1 > 1$ for each unit extremal of T_A, as listed in Equation 15, and similarly for extremals of T_B, as listed in Equation 16. Then Corollary 11 may be applied to conclude that the network must have stable aperiodic dynamics in T.

5.3. Example 3

We now deal with an example that fits into the framework of additive neural networks, i.e., Equations 1 and 2 with F_i given by Equation 3. The analysis in this example turns out to be more complicated but the method is the same and it illustrates how the method can sometimes be extended to situations where more than two cycles are involved.

The network is defined by Equations 1, 2 and 3, i.e.,

$$\dot{y} = -y + W\tilde{y} - \tau \tag{17}$$

with

$$\begin{aligned} W &= \begin{pmatrix} 0 & 10 & 6 & -2 \\ -4 & 0 & 1 & -5 \\ -8 & -8 & 0 & 3 \\ 1 & -2 & -9 & 0 \end{pmatrix}, \\ \tau &= (0.5, -3.5, -0.5, -0.5)'. \end{aligned}$$

Note that there is still no self-input in the network. The dynamics of this network can be captured by looking at cycles starting at and returning to the boundary between orthants 1100 and 1000, i.e., the boundary $(+, 0, -, -)$. Five cycles are possible on this boundary:

$$\begin{aligned} A &: 1000 \to 1001 \to 0001 \to 0011 \to 0010 \to 1010 \to 1000 \to 0000 \\ &\qquad \to 0100 \to 1100, \\ B &: 1000 \to 0000 \to 0100 \to 1100, \\ C &: 1000 \to 1001 \to 0001 \to 0011 \to 0010 \to 0110 \to 1110 \to 1100, \\ D &: 1000 \to 0000 \to 0001 \to 0011 \to 0010 \to 0110 \to 1110 \to 1100, \\ E &: 1000 \to 1001 \to 0001 \to 0011 \to 0010 \to 1010 \to 1110 \to 1100. \end{aligned}$$

The mappings and returning cones for each cycle can be calculated as before (again for details, see [16]). They are depicted in Figure 6. Let the mappings (Equation 6) for each cycle be called $M_i, i = A, B, C, D, E$, and the matrices in the mappings we denote A, B, C, D and E. Without giving all the details, we outline the procedure for identifying a trapping region and finding a common invariant proper cone. The entire octant $(+, -, -)$ is a trapping region, as it is the union of returning cones of the five cycle maps, and each cycle mapping M_A to M_E maps its returning cone back into this octant. The

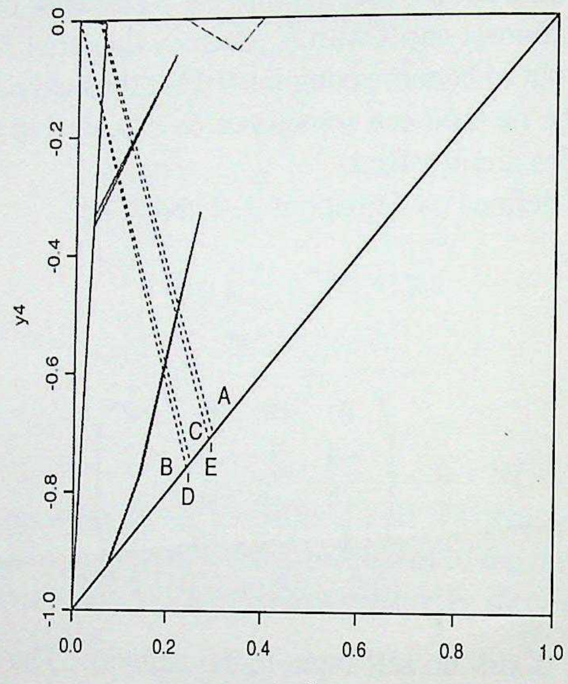

Figure 6. Trapping region in the $(+, 0, -, -)$ boundary, for the network of Equation 17, projected onto a plane orthogonal to vector $(1, -1, -1)'$, with only the y_1 and y_4 components plotted (i.e., x_1 and x_3 for $x \in \mathbb{R}^3$), so the entire $(+, -, -)$ octant of $\mathbb{R}^3$ is projected onto the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, -1)$. The dotted lines define the (projected) returning cones for the five cycles, which partition the octant: marked by the letters A to E. The solid lines show projections of the images of the returning cones under their respective mappings. The image of the B region is the leftmost triangle. The image of the A region is the narrow triangle connected to the B region image. The narrow region separated from these comprises the images of the C, D and E regions, where the E region maps to the short lower piece, the D region maps to the upper part of the longer piece, not extending into the returning cone for B and the C region maps to the rest of the longer piece, intersecting all 5 returning cones. We take our trapping region to be the union of the images of the B, C, D and E regions as well as the part of the image of the A region that lies in the B region. The dashed region at top shows the extent to which the invariant cone of the next figure encroaches into this octant.

images of the five returning cones under their respective mappings are also shown in Figure 6. Starting in this octant, after one mapping we must fall somewhere in the union of the images of the five regions, so this union is also a trapping region.

When we look for a common invariant proper cone, we find that we cannot work with the matrix A in cycle map M_A since its dominant eigenvalues are complex and these produce rotations in a plane through the origin that preclude any invariant cone being proper ($K \cap (-K) \neq \{0\}$). In fact it is an obvious consequence of Theorem 6 that if $\rho(A)$ is not an eigenvalue, A can leave no proper cone invariant. Thus we need to look at combinations in which cycle A can occur. We define a trapping region

$$T^* = M_B(C_B) \cup M_C(C_C) \cup M_D(C_D) \cup M_E(C_E) \cup (M_A(C_A) \cap C_B)$$

where C_i is the returning cone for cycle mapping M_i. To confirm that this is a trapping region, we need to verify that those parts of T^* lying in C_A are indeed mapped back into T^*, i.e., are mapped into $M_A(C_A) \cap C_B$. It is a matter of calculation to show that $M_A(M_C(C_C) \cap C_A) \in M_A(C_A) \cap C_B$, and likewise for $M_A(M_D(C_D) \cap C_A)$ and $M_A(M_B(C_B) \cap C_A)$. $M_E(C_E)$ does not intersect C_A. Now we observe that once in the trapping region, if we go through cycle A, it must be followed by cycle B, and it must be preceded either by cycle B, C or D. If it is preceded by cycle B, that is if we land in $M_B(C_B) \cap C_A$, then it turns out that the only part of C_B within the trapping region we could have come from is that part in $M_C(C_C)$, which can be verified by calculating the inverse image under M_B of $M_B(C_B) \cap C_A$. Thus, if we go through cycles B and A in succession, then the cycle before must have been C and the cycle after must be B again. In summary, the only combinations in which cycle A can occur are (C, A, B), (D, A, B), and (C, B, A, B). We will call these cycles F, G and H, so that $M_F = M_B \circ M_A \circ M_C$, for example. The matrices in these mappings will be called

$$F = BAC, G = BAD, H = BABC.$$

All trajectories through the trapping region, T^*, consist of compositions of the seven cycles B through H. Thus, the task is now to find a common invariant proper cone for the seven matrices B through H. As C, D and E have a negative dominant eigenvalue, we will actually use $-C$, $-D$ and $-E$. An appropriate invariant cone is shown projected onto the unit sphere in Figure 7. The cone is

$$K = \{k_1, k_0, CCk_0, k_2, k_3, -Ck_0, k_4\}^G,$$

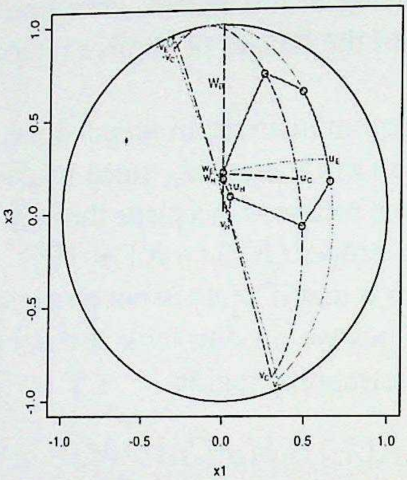

Figure 7. Invariant cone for the seven matrices in the Poincaré maps for the network of Equation 17, projected onto the unit sphere, indicated by solid lines. Projections of the invariant subspaces through eigenvectors are indicated by dashed lines.

where

$$
\begin{aligned}
k_0 &= u_{HC}, \text{ the dominant eigenvector of matrix } HC,\\
k_1 &= \{u_H, w_H\}^G \cap \{x_1 = 0\},\\
k_2 &= \{CCk_0, ECk_0\}^G \cap \{u_E, v_E\}^G,\\
k_3 &= \{-Ck_0, -Ek_0\}^G \cap \{u_E, -v_E\}^G,\\
k_4 &= \{u_E, w_E\}^G \cap \{x_1 = 0\}.
\end{aligned}
$$

In the figure, the generating points of the cones are marked by circles. The point k_1 is at the intersection of the line from w_H to u_H and the line marked W_B, which is the eigenspace for the matrix B, for which all three eigenvalues are 1. The rest are, counterclockwise from k_1: k_0, CCk_0, k_2, k_3, $-Ck_0$, k_4. It is clear that matrix $-C$ maps k_0 to $-Ck_0$ and $-Ck_0$ to CCk_0. Also, $-E$ maps k_0 to the boundary between $-Ck_0$ and k_3 and $-Ck_0$ to the boundary between CCk_0 and k_2. Furthermore k_2 and k_3 lie on a stable manifold of the dominant eigenvector of $-E$, namely u_E, and $-Ek_2$ and $-Ek_3$ lie on the boundary of the invariant cone between k_2 and k_3. B maps k_1 and k_4 to themselves and H maps k_0 to itself. Every other mapping of extremal vector for K by one of the seven matrices lands in the interior of K, as can be confirmed by direct calculation.

Careful inspection of the cases where an extremal point of K is mapped to the boundary of K shows that there are nevertheless few words W composed of our seven matrices that still map any point of K to its boundary. Clearly, B, $-C$, $-E$ and H do so, as discussed above. Figure 6 shows that $M_E(C_E) \in C_B$ entirely, so E can only be followed by B. B maps every point left on the boundary by $-E$ to $\text{int}(K)$, i.e., $-BE(K - \{0\}) \subseteq \text{int}(K)$. B maps k_1 and k_4 to themselves, but any other matrix maps these points to $\text{int}(K)$. Although $-C$ maps $-Ck_0$ to CCk_0, another extremal point, any subsequent multiplication takes this point to $\text{int}(K)$. Finally, k_0 is mapped by $-C$ to $-Ck_0$, an extremal point, and $-Ck_0$ is mapped by H back to k_0, by the definition of that point as an eigenvector of HC. Thus, aside from the four individual matrices above, the one other possible cycle that does not map K to $\text{int}(K)$ is $H(-C)$. Checking the eigenvectors of the matrices $-C$, $-E$, H and $-HC$ shows that only $-C$ has an eigenvector in its returning cone, and it is the second, not the dominant eigenvector. B, however, has a two-dimensional eigenspace associated with its one eigenvalue ($\mu = 1$), and has to be treated carefully. In principal, a matrix with dominant eigenvalue 1 can have invariant sets on which trajectories spiral in to the origin (or another fixed point on a coordinate subspace).

The matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix},$$

where $\alpha = \frac{2208}{133}$ and $\beta = \frac{408}{133}$, is lower triangular with all eigenvalues 1. Its eigenspace is the subspace $\{x_1 = 0\}$. Thus, although the Krein-Rutman theorem guarantees that an eigenvector for eigenvalue $\rho(B) = 1$ lies in K, it need not be unique (up to scalar multiples). In fact, here the entire two-dimensional eigenspace consists of eigenvectors corresponding to eigenvalue 1, and in particular, one boundary of the trapping region. If we start on this boundary of the trapping region where $x_1 = 0$, we stay on it, and thus continue to iterate the cycle mapping M_B, which will drive us in to the origin. However, there is no requirement that the trapping region be closed. We will therefore show that

$$T = T^* \cap \{x_1 > 0\}$$

is also a trapping region. No matrix other than B maps any point of T to $\{x_1 = 0\}$. Thus, we need only confirm that if we start away from $\{x_1 = 0\}$ in T, then in a finite number of iterations we leave B's returning cone, C_B, thus preventing convergence to $\{x_1 = 0\}$. Points in the interior of region

C_B in Figure 6 are mapped upwards, as well as closer to the boundary where $x_1(=y_1)=0$. To show that they must eventually cross the upper boundary of C_B, we first calculate that this boundary has slope $-22/7$. If we let $a=(a,-b,-c)'$, with $a,b,c\geq 0$, so that the ratio of the x_3 to x_1 components is $-c/a$, then

$$Ba=(a,-b-\alpha a,-c+\beta a)'$$

and the ratio of the x_3 to x_1 component is $-c/a+\beta$. Thus, each multiplication by B increases this ratio by more than 3. Any point in C_B away from $\{x_1=0\}$ has some fixed finite ratio of third to first components. Thus, only a finite number of multiplications by B are required to increase this ratio above $-22/7$ (but still below 0, since $\beta<22/7$). The trajectory of any such point thus starts with a finite sequence of B's before continuing with some other of our cycles, and it is not possible to converge to $\{x_1=0\}$. Thus, T is truly a trapping region, and B itself does not correspond to a periodic orbit through T or a stable spiral fixed point in T.

While the invariant cone, K, does encroach on the $(+,-,-)$ octant, the region of that octant that it includes is disjoint from the trapping region depicted in Figure 6. In fact, $T\cap(K\cup(-K))=\{0\}$. With the exceptions that have already been eliminated, any word, W made up of the seven cycle matrices, maps $K-\{0\}$ into its interior, so that Proposition 9 implies that no stable periodic orbit passes through T.

Finally, we need to apply Proposition 10, extended in a natural way to more than two mappings and regions, to show that there is no convergence to the origin. It is easiest to apply this to the five cycle matrices A through E (rather than working with the composite mappings F, G and H, and to check the extremals of the returning cones C_A through C_E rather than working with the smaller trapping region we used above. If convergence to the origin is impossible in the entire orthant boundary (i.e., $\mathcal{O}^{(0)}=C_A\cup C_B\cup C_C\cup C_D\cup C_E$) then it is certainly impossible in the trapping region, $T\subseteq\mathcal{O}^{(0)}$. However, the norms of the matrices multiplied by respective unit extremals are in this case not all >1. The two exceptions are $||Bu_B||_1=1$ and $||Bv_B||_1=1$, where $u_B=(0,-1,0)'$ and $v_B=(0,0,-1)'$. We know of course that convergence to the origin is possible on the plane $x_1=1$, but we showed above that trajectories cannot converge to this plane from inside T. A similar argument can be made here to extend Proposition 10 to cover this case. Note first that $Bu_B=u_B$ and $Bv_B=v_B$. While B may occur several times in succession, it was shown above that it must eventually be followed by one of the other cycle matrices along a trajectory starting in T. Clearly, $||AB^m u_B||_1 = ||Au_B||_1$ and similarly for the other matrices and for v_B. So, if these values are all > 1, we still cannot have convergence along trajectories from points in T. This is confirmed by calculations which we omit. Again, Corollary 11 is then invoked to conclude that there is an attractor in T on which the dynamics are aperiodic.

6. Discussion

We have presented a general method by which the existence of an attractor on which the dynamics is aperiodic may be proven analytically for a given Glass network. The class of Glass networks is quite broad, the functional form of the right-hand-sides of the equations being almost completely arbitrary, as long as the interactions between network units depend only on the output of the response function of other units. The form of the response is a step function but the indications are that the same behaviour will be found in networks where the response function is a steep sigmoid. Thus, the functional forms of many neural network models are encompassed in this class, including additive networks as well as those involving quadratic or higher order product terms. However, the results of this method will be less applicable to these networks with shallower sigmoids. Numerical work shows that in general as the slope of the sigmoid is decreased the complexity of the dynamics decreases [4, 5].

Within the class of Glass networks the method is quite general and the results we established above are valid for networks in any number of dimensions, n. However, in practice it may not be possible to apply the method, particularly for large n. Even when $n = 4$, there are examples for which the method has not so far proven successful, such as the example presented in [21, 16], due to the difficulty of finding (or perhaps the non-existence of) a common invariant proper cone for a large number of matrices.

The method of proof presented here depends on being able to find a non-trivial (fully n-dimensional) trapping region to begin with, and then being able to find a common invariant proper cone for all the cycle matrices of cycle maps defined on some part of the trapping region. The trick is to find a trapping region and common proper invariant cone that do not intersect. While this can sometimes be difficult, our Example 3 shows that there is some room for refinement in the method. While general results for these problems do not seem to exist in the matrix theory literature yet, good candidates for appropriate common invariant cones are ones constructed out of intersections of invariant subspaces through eigenvectors and the images of these subspaces under the matrices involved.

Also, our approach only guarantees stable aperiodic behaviour. It does not distinguish between different types of aperiodic behaviour, such as quasi-periodic and chaotic. Quasi-periodicity is very special and it seems certain that it is not present in the examples explored here, but further investigation into this distinction would be of interest.

While our method clearly has limitations, it would seem to be the first *general* approach to the analytic proof of *stable* complex behaviour in neural or other networks. Previous analytic work, as discussed in Section 1, relies on special structures built from known chaotic equations or systems.

While proof of the existence of stable complex activity in systems of differential equations is clearly of great mathematical interest, the utility of complicated dynamics in neural networks, whether biological or artificial, is still debated. It may be that complicated activity of any type (chaotic, quasi-periodic or very complicated periodic) is functionally equivalent, for example, if such activity is to act as a 'not recognized' response in an attractor neural network setting. On the other hand, the ergodicity of chaotic dynamics on an attractor may be an important distinction from other types of complicated activity, as argued for example by Skarda and Freeman [1] and by Kaneko [2]. In any case, it may well be important in applications to distinguish between transients and attractors–to know whether complicated activity is stable or is temporary and leads eventually to a stable limit cycle or fixed point. Even in applications, claims for chaos or complex dynamics based on numerical evidence is never completely satisfactory. The current work contributes to putting such claims on a rigorous foundation.

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Asymptotic Analysis of Noise-Amplified Oscillations for Subcritical Delays*

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Abstract

We develop a multi-scale analysis for stochastic differential delay equations with additive noise. Such models are particularly sensitive to noise when the system is near a critical point, such as a Hopf bifurcation, which marks a transition to oscillatory behavior. In particular, we are interested in the case when the combined effects of the noise and the delay amplify oscillations which would decay in the deterministic system. The derivation of reduced equations for the envelope of the oscillations provides an efficient analysis of the dynamics by separating the influence of the noise from the intrinsic oscillations over long time scales.

1. Introduction

In this paper we give an asymptotic analysis of the effect of noise on bifurcations due to delays. In particular, we are interested in the case where the sensitivity to the noise is demonstrated through a resonance. In the presence of noise, oscillations are sustained in the subcritical region, where oscillations die out in the absence of noise. In the absence of external periodic forcing this phenomenon has been called autonomous stochastic resonance [1], where

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the noise excites the oscillations intrinsic to the deterministic dynamics. It has been observed in many systems with delays, including models of neurons, lasers, and a variety of oscillators with delayed feedback (see [2]-[6], and references therein). It has also been observed in systems without delays, where it has been studied in the context of stochastic bifurcations [7].

For concreteness, we focus on the particular system

$$dx = -\alpha x(t) + c\frac{\theta^n}{\theta^n + x^n(t-\tau)} + \delta dw(t), \tag{1.1}$$

with $w(t)$ a standard Brownian motion. This is a canonical model of delayed negative feedback which has been used in the modeling of the pupil light reflex [8], [9]. There the effects of several different types of noise were compared through numerical simulations. One essential feature of this system is that in the deterministic case there is a Hopf bifurcation at a certain critical value of the delay τ_c. When there is no noise, oscillations about the steady state will decay in time when $\tau < \tau_c$, but they will not decay if $\tau > \tau_c$. This is a common behavior in models with delayed feedback based on delay differential equations [2]–[6], [10]–[13].

We focus on the subcritical case $\tau < \tau_c$ when the noise is small. In this case the dynamics are very sensitive to the noise, but the qualitative features of the stochastic dynamics are not dominated by the noise. Numerical simulations below show the typical effect of small noise for values of the delay near a Hopf bifurcation [7]. In the presence of small noise, oscillations with a frequency corresponding to the Hopf bifurcation of the deterministic system persist even for subcritical delays $\tau < \tau_c$. Without noise ($\delta = 0$), the amplitude of the oscillations would decay over time, and the system would approach the steady constant state. In the presence of noise, these oscillations are more prominent as the delay is increased, even though $\tau < \tau_c$ and the magnitude of the noise remains constant. Oscillations with the deterministic (natural) frequency are supported by the combination of the delay and the effect of the noise. While the deterministic frequency is the predominant feature in the oscillations, there are also small variations in the amplitude and the phase, due to the noise. These simulations show that there is an interaction of the deterministic and stochastic elements in the model.

Because of the large variability over time and over a range of parameters, a systematic numerical investigation of the system can be expensive. Therefore we look for an efficient method for analyzing the model, which isolates the effects of the noise from the natural oscillations in the system.

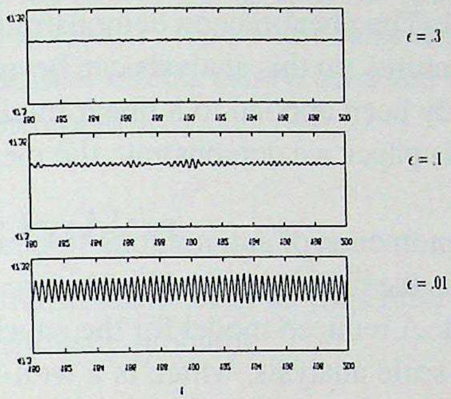


Figure 1. Simulation of equation (1.1) with $\delta = .001$, and $\tau = \tau_c - \epsilon^2$, with τ_c corresponding to Hopf bifurcation. Note that decreasing ϵ corresponds to approaching the critical delay. In the top graph the system is sufficiently far from the Hopf bifurcation point, so that the oscillations decay and the dynamics are essentially small random perturbations from the steady state. As ϵ decreases, the natural frequency of the system appears in the oscillations. For values of the delay close to the Hopf point, oscillations with the natural frequency of the deterministic system dominate the dynamics. There is also a small variation in amplitude and frequency due to the noise.

The difficulty that arises in analyzing these effects in stochastic delay differential equations is primarily related to the presence of the delay. Using an approach based on either a forward or backward Kolmogorov equation or even weak solutions of (1.1) [14], one is inevitably faced with the memory in the system due to the delay. For example, the Fokker-Planck (forward Kolmogorov) equation for the probability density is an integro-differential equation [15]. Below we outline an approach which avoids this complication.

In this paper we use information about the deterministic dynamics to obtain a reduced model in which the influence of the noise can be separated from the oscillations related to the bifurcation. This reduced model is a stochastic equation for the envelope of the oscillations, which varies on a long time scale. This approach has several advantages. The stochastic envelope equation allows a separation of the effect of the noise on the system over long times from the oscillations with deterministic frequency. The model is efficient for simulating long time behavior, since it describes the dynamics on large time scales. The reduced models have certain features which are useful for analytical solutions, based on small delay approximations [15] or linearized approximations, which we discuss in Section 3. The phenomenon demonstrated here occurs in many systems with similar features, so this analysis can be applied to a wide variety of models. It has already been applied to a linear model and a logistic model with delays [17]. In this paper we demonstrate the method in a more general nonlinear setting.

We focus on the phenomenon of autonomous stochastic resonance, considering parameter ranges in the vicinity of the Hopf bifurcation where small noise has a pronounced effect. A reduced model for the envelope of the oscillations is obtained via a multi-scale analysis, which is a well-known technique used near bifurcations in deterministic models [18], [19]. The method is combined with a projection onto the oscillatory modes of the linearized, deterministic system, which has similarities to Hale's reduction method [13]. Similar projection ideas have been used to reduce the stochastic Allen-Cahn equation to a system of stochastic differential equations for the coupled interfaces [16]. The analysis of this paper has similarities with multi-scale analysis of modulation equations in coupled oscillators [18] or patterns dynamics [19], but there are certain adaptations necessary for stochastic models. A discussion of this method for the supercritical case ($\tau > \tau_c$) in the context of the logistic equation with delay and its the relationship of the method to a Fourier series approximation is covered in [17].

The paper is organized as follows. In Section 2 we outline the multi-scale approach for deriving stochastic envelope equations for the oscillations in the subcritical case $\tau < \tau_c$. For clarity the technical details of this derivation are moved to the Appendix. In Section 3 we compare the stationary density of the process obtained from the approximation with that of the original process described by (1.1). We discuss these results and further applications of the method in Section 4.

2. Derivation of the Envelope Equations

In this section we outline the derivation of the stochastic envelope equations which gives an asymptotic approximation to the long time behavior of the noisy system (1.1) in the vicinity of the critical delay. We focus on the case where small noise $0 < \delta \ll 1$ amplifies the oscillations even though the delay is in the subcritical range $\tau < \tau_c$. In this model the combination of parameters c, θ, n and τ all play an important role in the dynamics. We focus in this paper on varying the length of the delay τ rather than the other parameters. Then we view the Hopf point as a value τ_c which of course depends on the values of the other parameters. Similar effects can be achieved by fixing the delay and varying the other parameters [9].

2.1. Deterministic Problem

The derivation via the multi-scale analysis uses information about the Hopf bifurcation in the deterministic problem,

$$\frac{dx}{dt} = -\alpha x(t) + c\frac{\theta^n}{\theta^n + x^n(t-\tau)}. \tag{2.1}$$

From this equation we determine the relationships between the critical delay τ_c, the natural frequency of the oscillations r, and the other parameters, α, c, θ, n. The oscillations are oscillations about a steady state x_0, which satisfies

$$\alpha x_0 = c\frac{\theta^n}{\theta^n + x_0^n}. \tag{2.2}$$

It is possible to determine α, c, θ, and n such that the steady state $x = x_0$ is stable for a given τ. In order to focus on oscillations about $x = x_0$, we rewrite the equation (2.1) in terms of $y = x - x_0$. Then (2.1) becomes

$$\frac{dy}{dt} = -\alpha y(t) + c\frac{\theta^n\left(x_0^n - [x_0 + y(t-\tau)]^n\right)}{(\theta^n + [x_0 + y(t-\tau)]^n)(\theta^n + x_0^n)}. \tag{2.3}$$

Next we determine an expression for the critical delay in the deterministic problem, corresponding to the Hopf bifurcation point, which will be useful in the calculations below. We linearize (2.3) about $y = 0$,

$$\frac{dy}{dt} \approx -\alpha y(t) - \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2}y(t-\tau), \tag{2.4}$$

and look for oscillatory solutions,

$$y(t) = e^{(\sigma + ir)t}. \tag{2.5}$$

That is, we look for a Hopf bifurcation from the $y = 0$ solution, which gives oscillations about $x = x_0$. Then, substituting (2.5) into (2.4) and using (2.2) yields equations for $\tau = \tau_c$ which must be satisfied in order to have purely oscillatory solutions to (2.4) ($\sigma = 0$),

$$-r \quad + \quad \frac{\alpha n x_0^n}{\theta^n + x_0^n} \sin r\tau_c = 0 \tag{2.6}$$

$$-\alpha \quad - \quad \frac{\alpha n x_0^n}{\theta^n + x_0^n} \cos r\tau_c = 0. \tag{2.7}$$

These equations relate the frequency r and the critical delay τ_c to the other parameters. It is a standard calculation to show that for $\tau > (<)\tau_c$, $\sigma > (<)0$.

2.2. Multiscale Analysis of the Noisy Nonlinear Problem

Here we use the method of multiple scales to derive stochastic equations which describe the envelope of the oscillations in (1.1). Since we are interested in oscillations about the deterministic steady state x_0, we rewrite (1.1) in terms of $y = x - x_0$,

$$dy = \left(-\alpha y(t) + c\frac{\theta^n(x_0^n - [x_0 + y(t-\tau)]^n)}{(\theta^n + [x_0 + y(t-\tau)]^n)(\theta^n + x_0^n)}\right)dt + \delta dw(t). \tag{2.8}$$

Now we apply the method of multi-scale analysis for delays near τ_c, that is, $\tau = \tau_c + \epsilon^2\tau_2$ for $\epsilon^2 \ll 1$ in the vicinity of the Hopf bifurcation point. Here we confine our attention to the case when $\tau < \tau_c$ ($\tau_2 < 0$), so that in the absence of noise, $x \to x_0$, $y \to 0$ as $t \to \infty$. The approach is very similar to the standard method of multiple scales used to analyze Hopf bifurcations in deterministic systems [18], [19]. We make appropriate adjustments for the stochastic terms.

We begin with the expansion for y,

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^2 y_2 + \cdots \\ &= \epsilon[A(T)\cos rt + B(T)\sin rt] + \epsilon^2[C(T)\cos 2rt \\ &\quad + D(T)\sin 2rt + E(T)] + \mathcal{O}(\epsilon^3), \end{aligned} \tag{2.9}$$

$$\tau = \tau_c + \epsilon^2\tau_2, \qquad \tau_2 < 0. \tag{2.10}$$

Here $T = \epsilon^2 t$ is a slow time, with the scale ϵ related to the proximity to the bifurcation point $\tau = \tau_c$. Thus the leading term in the expansion is simply an oscillatory function with a slowly varying amplitude or envelope, described by the functions $A(T)$ and $B(T)$. In order to capture the influence of the noise on the T time scale, we look for a solution (2.9) which is periodic on the fast time scale t, with an amplitude that varies stochastically over the long time scale; that is, A and B are stochastic quantities on the T time scale. The frequency of the oscillation is r, given by (2.7) with $\tau = \tau_c$. The functions $C(T)$, $D(T)$, $E(T)$ depend on $A(T)$ and $B(T)$ (4)–(4), and are determined in the analysis. In the following, we omit the argument T from $A(T)$, $B(T)$, $C(T)$, $D(T)$, and $E(T)$, except when the argument is at the delayed time $T - \epsilon^2\tau$.

We derive equations for the stochastic amplitudes A and B, assuming the following forms:

$$\begin{aligned} dA &= \psi_A dT + \sigma_A d\beta_A(T), \\ dB &= \psi_B dT + \sigma_B d\beta_B(T). \end{aligned} \tag{2.11}$$

Here β_A and β_B are two independent standard Brownian motions, and the coefficients ψ_A, ψ_B, σ_A, and σ_B are unknown and may depend on A and B. The assumed form of these equations will be shown to be consistent in the following analysis. We obtain these coefficients by deriving two equations for dy, first using Ito's formula, and then using the equation (1.1).

First, using Ito's formula we have

$$dy = \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial A}dA + \frac{\partial y}{\partial B}dB + \frac{\sigma_A^2}{2}\frac{\partial^2 y}{\partial A^2}dT + \frac{\sigma_B^2}{2}\frac{\partial^2 y}{\partial B^2}dT. \tag{2.12}$$

Combining (2.9) and (2.12) gives

$$\begin{aligned} dy = {} & \left(\epsilon\frac{\partial y_1}{dt} + \epsilon^2\frac{\partial y_2}{dt}\right)dt + \left[\epsilon\cos rt + \epsilon^2\frac{\partial y_2}{\partial A}\right] \\ & \times(\psi_A dT + \sigma_A d\beta_A(T)) \\ & + \left[\epsilon\sin rt + \epsilon^2\frac{\partial y_2}{\partial B}\right](\psi_B dT + \sigma_B d\beta_B(T)) \\ & + \epsilon^2\left(\frac{\sigma_A^2}{2}\frac{\partial^2 y_2}{\partial A^2} + \frac{\sigma_B^2}{2}\frac{\partial^2 y_2}{\partial B^2}\right)dT + \mathcal{O}(\epsilon^3). \end{aligned} \tag{2.13}$$

Note that in the spirit of multi-scale analysis, we treat functions of the slow time $T = \epsilon^2 t$ as independent of t.

Now we compare (2.13) to a second equation for dy, obtained by substitution of (2.9) into the right hand side of (2.8),

$$\begin{aligned} dy = -\Bigg[& \alpha\Big(\epsilon[A\cos rt + B\sin rt] + \mathcal{O}(\epsilon^2)\Big) \\ & + \frac{c\theta^n\Big(x_0^n - \Big(x_0+\epsilon[A(T-\epsilon^2\tau)\cos r(t-\tau_c)+B(T-\epsilon^2\tau)\sin r(t-\tau_c)]+\mathcal{O}(\epsilon^2)\Big)^n\Big)}{(\theta^n+(x_0+\epsilon[A(T-\epsilon^2\tau)\cos r(t-\tau_c)+B(T-\epsilon^2\tau)\sin r(t-\tau_c)]+\mathcal{O}(\epsilon^2))^n)(\theta^n+x_0^n)}\Bigg] dt + \delta dw. \end{aligned} \tag{2.14}$$

In the absence of noise ($\delta = 0$), a multi-scale analysis would proceed as follows. Using a perturbation expansion of y for $\epsilon \ll 1$ in (2.14), and setting the coefficients of like powers of ϵ equal to zero, yields a sequence of equations,

$$\mathcal{O}(\epsilon) : y_1'(t) + \alpha y_1(t) + \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2} y_1(t-\tau_c) = 0, \tag{2.15}$$

$$\begin{aligned} \mathcal{O}(\epsilon^j) : y_j'(t) + \alpha y_j(t) + \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2} y_j(t-\tau_c) \\ = R(y_1, y_2, \ldots y_{j-1}) \text{ for } j > 1. \end{aligned} \tag{2.16}$$

Equation (2.15) is simply the linear equation (2.4) which has the periodic solution $A\cos rt$ and $B\sin rt$. Allowing for a slowly varying modulation or envelope, we take $A(T)$ and $B(T)$ as functions of the slow time T. These functions are undetermined from (2.15), since T is treated as independent of t. The left hand side of (2.16) is the same linear operator as in (2.15) acting on y_j, and the right hand side depends on lower order terms y_k for $k < j$ in the expansion for y. In particular, it includes terms which involve $A(T)$ and $B(T)$, as well as $\cos mrt$ and $\sin mrt$ for m an integer. In order for (2.16) to have a solution, the right hand side must be orthogonal to the homogeneous solutions $\cos rt$ and $\sin rt$. Imposing these solvability conditions while treating the functions of the slow time $A(T)$ and $B(T)$ as independent of t, we obtain equations for $A(T)$ and $B(T)$. That is, $A(T)$ and $B(T)$ satisfy the equation which results from projecting the right hand side of (2.16) onto $\cos rt$ and $\sin rt$, while treating T as independent of t,

$$\int_0^{2\pi/r} R(y_1, y_2, \ldots y_{j-1}) \left\{ \begin{matrix} \cos rt \\ \sin rt \end{matrix} \right\} dt = 0. \tag{2.17}$$

The difference in the analysis of this paper is that we must also treat the noise. Then the procedure is as follows. We use a perturbation expansion for $\epsilon \ll 1$ for the drift in (2.14). Then we equate the two equations for dy, (2.13) and the expansion of (2.14). We equate separately the drift and diffusion coefficients in order to obtain the drift and diffusion coefficients in (2.11). As in the deterministic analysis, the projection onto $\cos rt$ and $\sin rt$ is used to determine these terms in the equations for A and B.

To treat the diffusion terms, we first rewrite the noise in (2.14),

$$\begin{aligned}\delta dw(t) &= \cos rt\ dw_1(t) + \sin rt\ dw_2(t)\\ &= \frac{\delta}{\epsilon}(\sin rt\, dw_1(\epsilon^2 t) + \cos rt\ dw_2(\epsilon^2 t))\\ &= \frac{\delta}{\epsilon}\cos rt\ dw_1(T) + \frac{\delta}{\epsilon}\sin rt\ dw_2(T),\end{aligned} \tag{2.18}$$

using well-known properties of white noise [20], with w_1 and w_2 two independent standard Brownian motions. The diffusion terms in (2.13) are

$$[\epsilon\cos rt + \mathcal{O}(\epsilon^2)]\sigma_A d\beta_A(T) + [\epsilon \sin rt + \mathcal{O}(\epsilon^2)]\sigma_B d\beta_B(T). \tag{2.19}$$

Then we neglect the $\mathcal{O}(\epsilon^2)$ terms and equate the noise terms from both equations (2.14) and (2.13) for dy, as given in (2.18) and (2.19), to get the leading order expressions for σ_A and σ_B,

$$\begin{aligned}\epsilon\cos rt\ \sigma_A d\beta_A(T) + \epsilon\sin rt\ \sigma_B d\beta_B(T) \sim \frac{\delta}{\epsilon}(&\cos rt\ dw_1(T)\\ &+ \sin rt\ dw_2(T)).\end{aligned} \tag{2.20}$$

Now we treat this equation from the viewpoint of the method of multiple scales, where the two time scales t and T are viewed as independent, and we use the the orthogonality of the modes $\sin jrt$ and $\cos jrt$. Here we use the inner products

$$\begin{aligned}\int_0^{2\pi/r}\Big[&\cos rt\Big(\epsilon\sigma_A d\beta_A(T) - \frac{\delta}{\epsilon}dw_1(T)\Big)\\ &+ \sin rt\Big(\epsilon\sigma_B d\beta_B(T) - \frac{\delta}{\epsilon}dw_2(T)\Big)\Big]\left\{\begin{array}{c}\cos rt\\ \sin rt\end{array}\right\}dt = 0,\end{aligned} \tag{2.21}$$

and integrate with respect to the fast time scale, treating the terms which depend on the slow time scale T as independent of t in this integration. Thus we identify $\beta_A(T) = dw_1(T)$ and $\beta_B(T) = dw_2(T)$, which then yields

$$\sigma_A = \sigma_B = \frac{\delta}{\epsilon^2}. \tag{2.22}$$

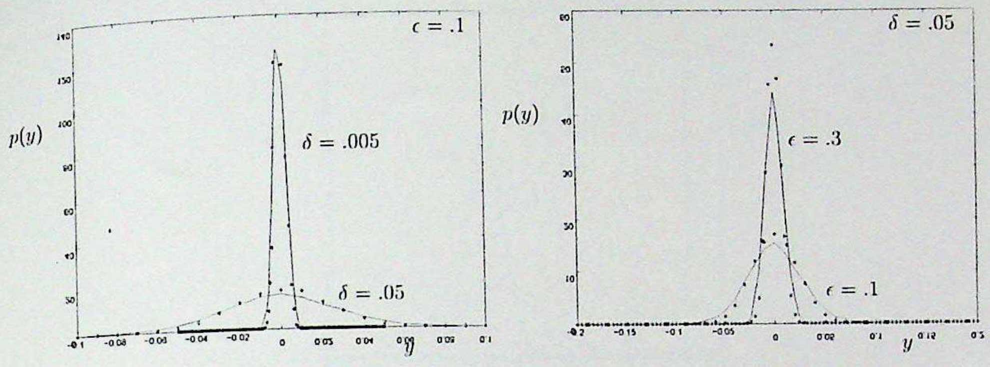
For the drift, essentially we follow the procedure outlined for the deterministic case in (2.15)-(2.17) after equating the drift terms in (2.14) and (2.13). We collect terms of like powers of ϵ and set them equal to zero. In the Appendix we show that there is cancellation up to $\mathcal{O}(\epsilon^3)$ in the drift for (2.14), using the definitions of r and τ_c, and the coefficients C, D, and E (4)-(4). Then we take the inner product of the $\mathcal{O}(\epsilon^3)$ terms with $\cos rt$ and $\sin rt$, which yields two equations which involve ψ_A, ψ_B in terms of A and B and the other parameters in the problem. The details of the analysis appear in the Appendix. Then the equations are

$$\begin{aligned}
\psi_A &= b_1\Big[-\frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}\sin r\tau_c+\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2}\cos r\tau_c \\
&\qquad -r\tau_2\Big(B(T)\cos r\tau_c+A(T)\sin r\tau_c\Big)\Big] \\
&\quad +\frac{b_2}{2}[(AC+DB)\cos r\tau_c+(BC-AD)\sin r\tau_c+EA\cos r\tau_c \\
&\qquad -EB\sin r\tau_c] \\
&\quad +\frac{b_3(A^2+B^2)}{8}[A\cos r\tau_c-B\sin r\tau_c]+\mathcal{O}(\epsilon) \qquad (2.23) \\
\psi_B &= b_1\Big[\frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}\cos r\tau_c+\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2}\sin r\tau_c \\
&\qquad -r\tau_2\Big(B(T)\sin r\tau_c-A(T)\cos r\tau_c\Big)\Big] \\
&\quad +\frac{b_2}{2}[(AC+DB)\sin r\tau_c-(BC-AD)\cos r\tau_c \\
&\qquad +EA\sin r\tau_c+EB\cos r\tau_c] \\
&\quad +\frac{b_3(A^2+B^2)}{8}[A\sin r\tau_c+B\cos r\tau_c]+\mathcal{O}(\epsilon) \qquad (2.24)
\end{aligned}$$

Neglecting the $\mathcal{O}(\epsilon)$ terms in these expressions, we combine them with (2.22) to get the coefficients in (2.11).

3. Comparison of the Original Model with the Multi-Scale Approximation

In the following figures we illustrate the performance of the multi-scale approximation (2.9) for $y = x - x_0$ by comparison with the solution to the original



(a) (b)

Figure 2. Comparison of the densities obtained from (2.9) (*'s) and (1.1) (solid lines) for (a) $\epsilon = .1$, and $\delta = .005, .05$ (*'s) and (b) $\epsilon = .1, .3$, and $\delta = .05$ (*'s). Note that the variance of the process increases as the noise increases, and as the proximity to the critical delay decreases.

model (1.1). The choice of parameter values is $n = 21$, $\theta = 50$, $c = 200$, $\alpha = 3.21$. With these parameter values, $x_0 \approx 47.332$, $r \approx 15.8775$ and $\tau_c \approx .1115$. This choice of parameter values is similar to that used in [9]. While it may appear that the critical delay τ_c is small in this case, in fact, τ_c is not small relative to the period of the oscillations. That is, using a rescaled time $\tilde{t} = t/\tau_c$, we get a critical delay of $\tilde{\tau}_c = 1$, and a natural frequency of $\tilde{r} \approx 1.7703$.

We compare results from realizations for times sufficiently long so that the process follows the stationary density. From 5000 realizations of (1.1) we approximate the stationary density of $y = x - x_0$, and from 5000 realizations of (2.11) we approximate the stationary density using the multiscale approximation $y \sim A\cos rt + B\sin rt$, for large t. These approximations of the stationary density of y are compared in the following graphs. Initial conditions are such that the process starts near $x = x_0$, or $A = B = 0$.

In Figure 2a, we compare the behavior for $\epsilon = .1$ with $\delta = .005$ and $\delta = .05$. As expected, as the noise increases, the variance of the process increases and the concentration of the density decreases. In Figure 2b we compare the behavior for $\epsilon = .1$ and $\epsilon = .3$ for $\delta = .05$. As τ approaches τ_c, ϵ decreases, and the oscillations increase as the noise causes a resonance. Then the variance of the process increases.

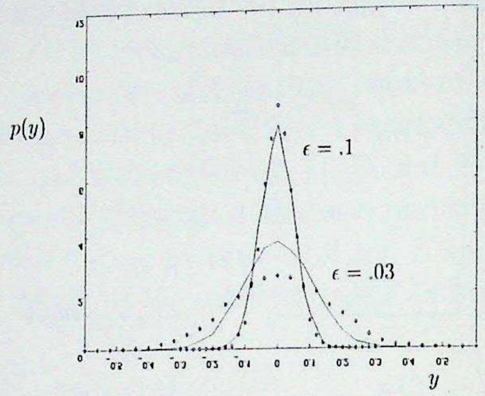


Figure 3. Comparison of the densities obtained from (2.9) ($*$'s) and (1.1) (solid lines) for $\delta = .1$ and $\epsilon = .1$ and $\epsilon = .03$, values for which one might expect that the asymptotic approximation would break down. For $\epsilon = .1$, the multi-scale approximation gives a good approximation, but as ϵ decreases the multi-scale approximation breaks down, as it describes a process with a variance which is too large.

In both cases in Figure 2b), δ/ϵ^2, the diffusion coefficient indicating the effective magnitude of the noise in the envelope equations (2.11), is not small. Figure 3 provides a numerical exploration to test the range of validity of the multi-scale approximation. It gives a good approximation if $\delta/\epsilon \leq 1$, but it gives a variance which is too large if δ/ϵ exceeds unity. In order to get reasonable results from these extreme values of ϵ and δ, we rewrite $\tau = \tau_c - \epsilon^2\tau_2$, as $\tau_c - \hat{\epsilon}^2\hat{\tau}_2$, choosing $\hat{\epsilon}$ so that $\delta/\hat{\epsilon} < 1$. Under such a rescaling, there is a balance between drift and diffusion in the envelope equations, so that the multi-scale approximation gives a good approximation.

The fact that the asymptotic analysis provides a good approximation for a large range of parameters is due to the dissipative behavior of the drift terms in the equations for A and B. Then, for values of δ and ϵ for which the effective magnitude noise in (2.11) is not too large, variations away from $A = B = 0$ do not dominate the dynamics since the damping in the drift drives the system towards zero amplitude variation. In Figure 4 we demonstrate that for $\tau < \tau_c$, the linear approximation to (2.11) gives a good approximation to the behavior of A and B, and we discuss the asymptotic validity in the context of that approximation.

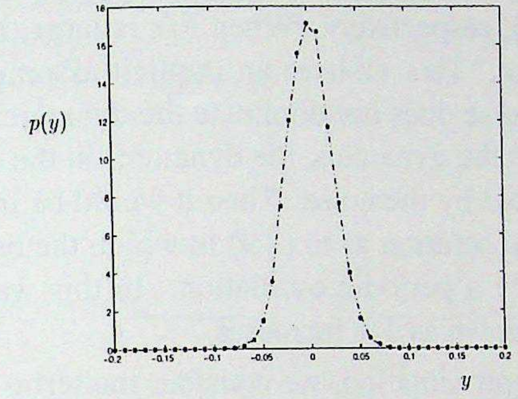


Figure 4. Comparison of the densities obtained for $y = A\cos rt + B\sin rt$, with A and B given by the linear approximation (3.1)-(3.2) (dashdotted), and the nonlinear equations (2.11) (*'s), for $\delta = .05$, $\epsilon = .1$

Neglecting the nonlinear terms in the equation for A and B,

$$\begin{aligned} dA &\approx b_1\Big[-\frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}\sin r\tau_c+\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2}\cos r\tau_c \\ &\quad -r\tau_2\Big(B(T)\cos r\tau_c+A(T)\sin r\tau_c\Big)\Big]dT+\frac{\delta}{\epsilon^2}d\beta_1(T), \qquad (3.1) \\ dB &\approx b_1\Big[\frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}\cos r\tau_c+\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2}\sin r\tau_c \\ &\quad -r\tau_2\Big(B(T)\sin r\tau_c-A(T)\cos r\tau_c\Big)\Big]dT+\frac{\delta}{\epsilon^2}d\beta_2(T). \qquad (3.2) \end{aligned}$$

In Figure 4 simulations using the linearized approximations to the envelope equations (3.1)-(3.2) are compared with simulations using the original system (1.1). The good agreement demonstrates that the linear terms in ψ_A and ψ_B dominate the dynamics of the slow envelope described by A and B.

From the linearized equations we can see how the ratio $\delta/\epsilon < 1$ naturally occurs as a criterion for the asymptotic validity of the multi-scale approximation. Replacing $\epsilon A = a$ and $\epsilon B = b$ in the linearized approximation (3.1)-(3.2) results in equations for a and b

$$\begin{aligned} da &\approx \psi_a dT + \frac{\delta}{\epsilon}d\beta_A(T) \\ db &\approx \psi_b dT + \frac{\delta}{\epsilon}d\beta_B(T). \end{aligned} \qquad (3.3)$$

Here ψ_a and ψ_b are just the drift coefficients from (3.1)-(3.2) with A and B replaced by a and b, respectively. When δ/ϵ is large, the system (3.3) is dominated by the noise. This violates an implicit assumption in the multi-scale expansion: the noise does not dominate the dynamics. If the noise was the dominant feature in the dynamics, the dynamics in the original variable x would also be dominated by the noise. Then it would be incorrect to assume an approximate form of solution as in (2.9) in which the behavior on the fast time scale t was simply a periodic oscillation. In this way, the multi-scale approximation breaks down as δ/ϵ increases.

As an additional approximation, we consider the terms in (3.1) and (3.2) which are at the delayed time $(T - \epsilon^2\tau)$. Since $\epsilon^2\tau$ is a small delay, well below any possible critical delay which would result in oscillations for A and B in the absence of noise, we consider the approximation which results from neglecting the delay $\epsilon^2\tau$ for $\epsilon^2 \ll 1$ in (3.1)-(3.2). Then the equations for the approximations $\widehat{A}$ and $\widehat{B}$ are

$$d\widehat{A} = \left[-b_1 r\tau_2(\widehat{B}(T)\cos r\tau_c + \widehat{A}(T)\sin r\tau_c)\right] dT + \frac{\delta}{\epsilon^2} d\beta_A(T) \quad (3.4)$$

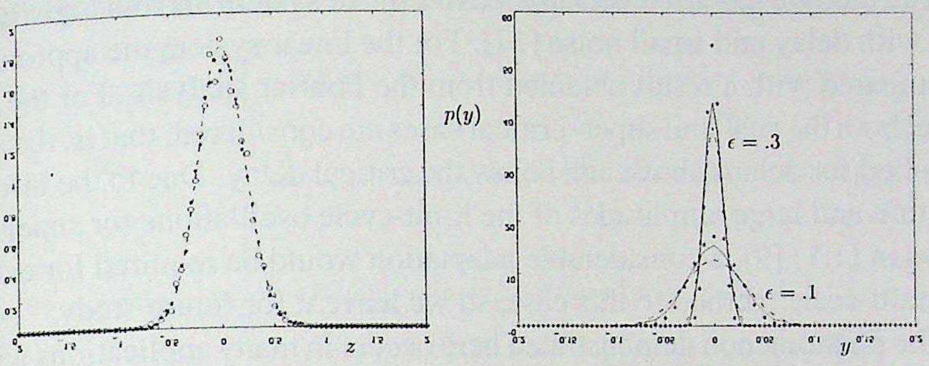
$$d\widehat{B} = \left[-b_1 r\tau_2(\widehat{B}(T)\sin r\tau_c - \widehat{A}(T)\cos r\tau_c)\right] dT + \frac{\delta}{\epsilon^2} d\beta_B(T). \quad (3.5)$$

This coupled system describes a two-variable Ornstein-Uhlenbeck process, where the stationary density is normal with the following properties,

$$\mathbf{E}[\widehat{A}] = \mathbf{E}[\widehat{B}] = \mathbf{E}[\widehat{AB}] = 0, \quad (3.6)$$

$$\mathbf{E}[\epsilon^2 \widehat{A}^2] = \mathbf{E}[\epsilon^2 \widehat{B}^2] = \sigma \equiv \frac{\delta^2}{2\epsilon^2 b_1 r\tau_2 \sin r\tau_c}. \quad (3.7)$$

Then in the steady state $\widehat{A}$ and $\widehat{B}$ are two independent normal random variables which are identically distributed as $\mathcal{N}(0,\sigma)$. Furthermore, $y = \widehat{A}\cos rt + \widehat{B}\sin rt$ is a normal random variable with zero mean and variance 2σ. In Figure 5 we compare the stationary density of A, B and y obtained from the full multi-scale approximation (2.23)-(2.24) with the density of the two variable Ornstein-Uhlenbeck process described by (3.4)-(3.5). The approximation does surprisingly well, even though the nonlinear terms and the delay in the envelope equations have been neglected.



(a) (b)

Figure 5. (a) Comparison of the steady state density obtained from simulation of (2.11) for A ($*$'s) and B (o's) with the normal density $p(z) = (2\pi\sigma)^{-1/2}e^{-z^2/(2\sigma)}$ (dash-dotted line) with σ given by (3.7). Here $\delta = .05$, $\epsilon = .1$. (b) Comparison of the steady state density of $x - x_0$ obtained from (1.1) (solid line), y obtained from the multi-scale approximation using (2.11) ($*$'s), and the normal density $p(y) = (4\pi\sigma)^{-1/2}e^{-y^2/(4\sigma)}$ (dash-dotted line) for $\delta = .05, \epsilon = .1$ and $\epsilon = .3$. These are the same parameter values as in Figure 2.

4. Discussion

We apply a multiscale analysis to analyze the effect of noise on a model with delayed negative feedback. This model is particularly sensitive to noise when the delay is near a critical value, corresponding to a Hopf bifurcation. In the subcritical case, oscillations which would decay in the absence of noise are amplified by the interaction of the noise and the delay. We derive stochastic equations for the slow evolution of the envelope of these oscillations at the Hopf frequency. The analysis leads to a description which separates the deterministic and stochastic elements of the dynamics. Advantages of this approximation include efficiency of simulations over large times and relatively simple dynamics described by the envelope equations.

The asymptotic validity of the approximation is explored by comparison with the original model. Since the drift in the stochastic envelope equations is dissipative, certain approximations are valid in a significant range of parameters. One particularly simple approximation which performs well is the linearized system for the envelope equations. In addition, since the delay in the envelope equation is very small, approximations which eliminate the delay can also be used. This leads to an analytical approximation for the behavior of system.

The method has also been applied to a linear system and the logistic model, both with delay and small noise [17]. For the linear system the approximation is compared with a result obtained from the Fourier analysis. For the logistic model both the sub- and super- critical cases are considered; that is, the analysis is applied for delays above and below the critical delay. Due to the bifurcation structure and large amplitudes of the limit-cycle oscillations for super-critical delays in (1.1) [9], a considerable adaptation would be required for extending the multi-scale method to this case, so we leave it for future study.

The phenomenon demonstrated here occurs in many applications with similar features, including neuronal models, optics, and information systems [2]-[6]. The analysis exploits the interaction of the noise with the Hopf bifurcation, features which are common to these other models. Thus the analysis can be applied to a wide variety of stochastic models, including those without delays. The method is particularly useful for systems with delays since it avoids complications due to the memory in the system, which can make other analyses impractical.

A. Appendix

We start with the two equations for dy obtained from Ito's formula (2.13) and (2.14) obtained by substitution of (2.9) into (2.8). We derive the nonlinear evolution equations for the subcritical $\tau < \tau_c$ ($\tau_2 < 0$) case. Since we are interested in the delay-related bifurcation from $y = 0$, a Taylor series expansion about $y = 0$ of the nonlinear term on the right hand side of the equation (2.14) is useful,

$$\begin{aligned}
& c\frac{\theta^n(x_0^n - (x_0 + y(t-\tau))^n)}{(\theta^n + (x_0 + y(t-\tau))^n)(\theta^n + x_0^n)} \\
&= b_1 y(t-\tau) + \frac{b_2}{2!}y(t-\tau)^2 + \frac{b_3}{3!}y(t-\tau)^3 + \cdots \qquad \text{(A.1)}
\end{aligned}$$

$$b_1 = -\frac{n\alpha x_0^n}{\theta^n + x_0^n} \qquad \text{(A.2)}$$

$$b_2 = -\frac{n(n-1)\alpha x_0^{n-1}}{\theta^n + x_0^n} + \frac{2n^2\alpha x_0^{2n-1}}{(\theta^n + x_0^n)^2} \qquad \text{(A.3)}$$

$$\begin{aligned}
b_3 &= -\frac{n(n-1)(n-2)\alpha x_0^{n-2}}{\theta^n + x_0^n} + \frac{6n^2(n-1)\alpha x_0^{2n-2}}{(\theta^n + x_0^n)^2} \\
&\quad -\frac{6n^3\alpha x_0^{3n-2}}{(\theta^n + x_0^n)^3} \qquad \text{(A.4)}
\end{aligned}$$

Here we have used the expression for x_0 (2.2). Furthermore, we take

$$\begin{aligned}
C(T) &= \frac{b_2}{2(4r^2+\alpha)}\Bigg(-2r[(A^2-B^2)\sin r\tau_c \cos r\tau_c + AB(\cos^2 r\tau_c - \sin^2 r\tau_c)] \\
&\qquad + \alpha\left[-2AB\sin r\tau_c\cos r\tau_c + \frac{A^2-B^2}{2}(\cos^2 r\tau_c - \sin^2 r\tau_c)\right]\Bigg) \\
D(T) &= \tfrac{b_2}{2(4r^2+\alpha)}\Bigg(\alpha[(A^2-B^2)\sin r\tau_c \cos r\tau_c + AB(\cos^2 r\tau_c - \sin^2 r\tau_c)] \\
&\qquad + 2r\left[-2AB\sin r\tau_c\cos r\tau_c + \tfrac{A^2-B^2}{2}(\cos^2 r\tau_c - \sin^2 r\tau_c)\right]\Bigg)
\end{aligned} \tag{A.5}$$

$$E(T) = \frac{b_2}{4\alpha}(A^2+B^2). \tag{A.6}$$

This choice leads to some cancellation, which we describe below. From Ito's formula (2.12) we have:

$$\begin{aligned}
dy &= \epsilon(-rA(T)\sin rt + rB(T)\cos rt - 2\epsilon r C(T)\sin 2rt \\
&\quad +2\epsilon r D(T)\cos 2rt + \mathcal{O}(\epsilon^2))dt \\
&\quad + \left[\epsilon\cos rt + \epsilon^2\left(\frac{\partial C}{\partial A}\cos 2rt + \frac{\partial D}{\partial A}\sin 2rt + \frac{\partial E}{\partial A}\right) + \mathcal{O}(\epsilon^3)\right] \\
&\qquad \times(\psi_A dT + \sigma_A d\beta_A(T)) \\
&\quad + \left[\epsilon\sin rt + \epsilon^2\left(\frac{\partial C}{\partial B}\cos 2rt + \frac{\partial D}{\partial B}\sin 2rt + \frac{\partial E}{\partial B}\right) + \mathcal{O}(\epsilon^3)\right] \\
&\qquad \times(\psi_B dT + \sigma_B d\beta_B(T)) \\
&\quad +\epsilon^2\left(\frac{\sigma_A^2}{2}\left[\frac{\partial^2 C}{\partial A^2}\cos 2rt + \frac{\partial^2 D}{\partial A^2}\sin 2rt + \frac{\partial^2 E}{\partial A^2} + \mathcal{O}(\epsilon)\right]\right. \\
&\qquad \left. +\frac{\sigma_B^2}{2}\left[\frac{\partial^2 C}{\partial B^2}\cos 2rt + \frac{\partial^2 D}{\partial B^2}\sin 2rt + \frac{\partial^2 E}{\partial B^2} + \mathcal{O}(\epsilon)\right]\right) dT
\end{aligned} \tag{A.7}$$

and from substitution of (A.1) and (2.9) into (2.14) we have

$$\begin{aligned}
dy &= -\epsilon\alpha\,[A\sin rt - B\cos rt] \\
&\quad + \epsilon b_1\,(A(T)\cos r(t-\tau_c) + B(T)\sin r(t-\tau_c)) \\
&\quad - \epsilon^2\left(\alpha\,[C\cos 2rt + D\sin 2rt + E] - \frac{b_2}{2}(A\cos rt + B\sin rt)^2\right) \\
&\quad + \epsilon^3\,\Big[rb_1\tau_2(A\sin r(t-\tau_c) - B\cos r(t-\tau_c))
\end{aligned}$$

$$
\begin{aligned}
&+ b_1 \bigg(\frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \cos r(t-\tau_c) \\
&+ \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \sin r(t-\tau_c) \bigg) \\
&+ b_2 \left(A \cos r(t-\tau_c) + B \cos r(t-\tau_c)\right) \left(C \cos 2r(t-\tau_c)\right. \\
&\left. + D \sin 2r(t-\tau_c) + E\right) \\
&+ \left. \frac{b_3}{3!} \left(A \cos r(t-\tau_c) + B \sin r(t-\tau_c)\right)^3 \right] dt \\
+ \mathcal{O}(\epsilon^4) + \delta dw(t). & \qquad\qquad (A.8)
\end{aligned}
$$

Here we have used (A.1) for $\epsilon \ll 1$, and we have also rewritten

$$
\begin{aligned}
A(T-\epsilon^2\tau) &= A(T) + \epsilon^2 \tfrac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2} \\
B(T-\epsilon^2\tau) &= B(T) + \epsilon^2 \tfrac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}
\end{aligned}
\qquad (A.9)
$$

and have assumed that

$$
\frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} = \mathcal{O}(1), \;\; \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} = \mathcal{O}(1) \qquad (A.10)
$$

Then we equate (A.7) and (A.8) and substitute (A.5), (A.6) into (A.8). This results in the cancellation of the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms, using the definitions of r and τ_c (2.7), and C, D, and E from (A.5), (A.6).

The remaining terms are then

$$
\begin{aligned}
& \epsilon \sin rt(\psi_B dT + \sigma_B d\beta_B(T)) + \epsilon \cos rt(\psi_A dT + \sigma_A d\beta_A(T)) \\
& + \epsilon^2 \bigg(\frac{\sigma_A^2}{2} \left[\frac{\partial^2 C}{\partial A^2} \cos 2rt + \frac{\partial^2 D}{\partial A^2} \sin 2rt + \frac{\partial^2 E}{\partial A^2} \right] \\
& \qquad + \frac{\sigma_B^2}{2} \left[\frac{\partial^2 C}{\partial B^2} \cos 2rt + \frac{\partial^2 D}{\partial B^2} \sin 2rt + \frac{\partial^2 E}{\partial B^2} \right] \bigg) dT \\
& + \text{other modes and higher order terms} \\
= & \bigg(\epsilon^3 \Big[r b_1 \tau_2 (A \sin r(t-\tau_c) - B \cos r(t-\tau_c)) \qquad (A.11) \\
& \quad + b_1 \bigg(\frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \cos r(t-\tau_c) \\
& \qquad + \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \sin r(t-\tau_c) \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+b_2\left(A\cos r(t-\tau_c)+B\cos r(t-\tau_c)\right)\\
&\times\left(C\cos 2r(t-\tau_c)+D\sin 2r(t-\tau_c)+E\right)\\
&+\frac{b_3}{3!}\left(A\cos r(t-\tau_c)+B\sin r(t-\tau_c)\right)^3\Bigg]+\mathcal{O}(\epsilon^4)\Bigg)dt+\delta dw(t)
\end{aligned}
$$

The slowly varying functions $C(T-\epsilon^2\tau)$, $D(T-\epsilon^2\tau)$, and $E(T-\epsilon^2\tau)$ are rewritten as in (A.9), so that these functions with delayed arguments are included in the $\mathcal{O}(\epsilon^4)$ terms.

As described at the end of Section 2, we equate noise terms and drift terms separately, which leads to σ_A and σ_B is given in (2.18)-(2.22). For the drift terms we again use the projection as in (2.21). This is equivalent to equating the coefficients of both $\cos rt$ and $\sin rt$, and neglecting all other modes and higher order terms. For example, the terms

$$
\begin{aligned}
+\epsilon^2\Bigg(&\frac{\sigma_A^2}{2}\left[\frac{\partial^2 C}{\partial A^2}\cos 2rt+\frac{\partial^2 D}{\partial A^2}\sin 2rt+\frac{\partial^2 E}{\partial A^2}\right]\\
&+\frac{\sigma_B^2}{2}\left[\frac{\partial^2 C}{\partial B^2}\cos 2rt+\frac{\partial^2 D}{\partial B^2}\sin 2rt+\frac{\partial^2 E}{\partial B^2}\right]\Bigg)dT,
\end{aligned}
$$

do not play a role since they contain terms which are orthogonal to the primary modes $\sin rt$ and $\cos rt$ on the t time scale. Then we are left with equations for ψ_A and ψ_B, which yield (2.23)-(2.24).

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Stability Analysis of Rings of Pulse-Coupled Oscillators: The Effect of Phase Resetting in the Second Cycle after the Pulse is Important at Synchrony and for Long Pulses

S. A. Oprisan and C. C. Canavier

Abstract

Rings of limit cycle oscillators coupled in a pulsatile manner to simulate the chemical synaptic coupling of neural oscillators are approximated by discrete maps based on a phase resetting analysis. Linearization about a fixed pattern of phase resetting is employed to predict the stability of patterns such as synchrony or alternation. Novel results are obtained with respect to synchrony in a two-neuron ring, and previous results regarding nonsynchronous firing patterns are extended to include the effects of phase resetting on two consecutive cycles, rather than just one. The effects on two cycles were considered because we wish to apply our results so circuits in which the pulses last a significant fraction of the cycle, and can easily affect two cycles if they occur late in the cycle.

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[0] AMS (MOS) 2000 Subject classifications: 37J25, 92B20.

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1. Introduction

The collective activity of coupled neural oscillators is of interest in several contexts, including the study of central pattern generation (CPGs) for rhythmic motor activity such as locomotion [16], and in studies of the oscillations that are thought to underlie epileptic foci, generalized epileptic seizures, and the various normal EEG rhythms, as well as perception and even consciousness [14, 19]. Neural oscillators are complex and nonlinear, and their equations of state are not readily available. However, a phase resetting curve (PRC) is easily generated experimentally by tabulating the effect on the cycle period produced by a perturbation as a function of the temporal phase (ϕ), or elapsed time normalized by the cycle period, at which it is applied. Phase resetting curves are usually generated in an open loop condition, that is, the oscillator is isolated from the circuit when the test perturbations are applied. In neural circuits with pulsatile coupling, the form of the coupling is intended to simulate a chemical synapse, which is inherently directional in that the postsynaptic neuron is perturbed only when the presynaptic neuron is above threshold. In order to apply PRC data to the analysis of neural circuits, the perturbation used to generate the PRC should approximate the perturbation received when the circuit is complete, that is, the closed loop condition. Given the assumption that the open loop perturbation approximates the closed loop perturbation, two powerful mathematical tools can be employed for the analysis of circuits composed of neurons characterized by their phase resetting curves: (1) the systematic tabulation of periodicity constraints and (2) the linearization about a repetitive pattern of phase resetting. The first tool can decipher whether a particular pattern can be produced in the absence of any infinitesimal amount of noise, whereas the second can provide stability information regarding the robustness of such a pattern in the presence of noise or perturbations. In order to to predict oscillatory patterns exhibited by neural circuits composed of physiological neurons, using only data that is readily obtainable from the isolated component neurons, we have previously developed a method based on several assumptions: (1) all component neurons in the circuit are endogenous limit cycle oscillators, (2) there are no synaptic delays, (3) each neuron receives one and only one perturbation, called a synaptic input, per cycle (hence the ring configuration), (4) each neuron returns to its unperturbed limit cycle before the next input is received and (5) all oscillators are entrained in a 1:1 mode at a common frequency. Under these restrictive assumptions, we can accurately

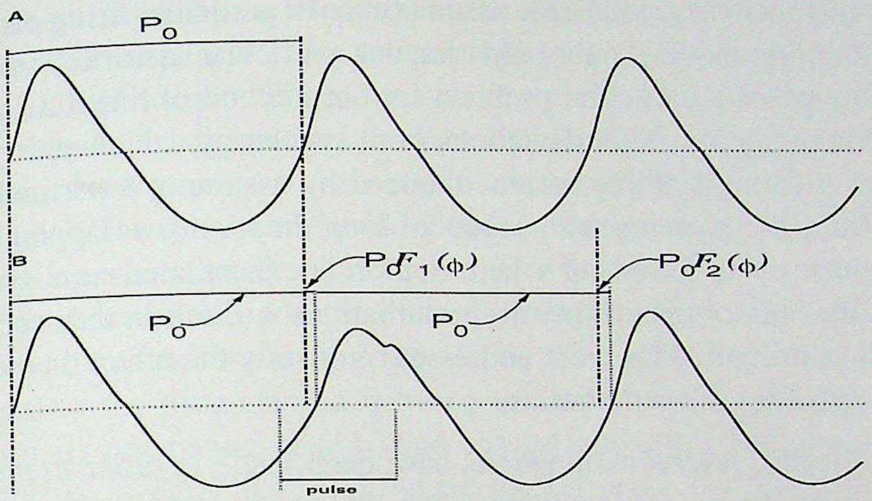

Figure 1. Phase Response Curve Generation. A. Free running oscillation showing unperturbed intrinsic period P_0. B. Example of a long perturbation that affects two burst cycles. The effect of the perturbation on the period of the first burst is quantified by F_1 and on the second burst by F_2..

predict the oscillatory modes exhibited by circuits composed of model neurons [2]. Our previous analyses of stability [3] treated pulsatile coupling as a set of first order maps, in which only the timing of the most recent perturbation determined the cycle length in which the perturbation occurs for each component oscillator; this paper extends that work to include a set of second order maps instead, in order to properly account for the effects of pulses that occur late in the cycle, for long pulses especially.

Biological circuits are often composed of burst firing neurons, so a single input, or perturbation, to a neuron within a circuit can be characterized as a burst, which generally lasts for a significant fraction of the intrinsic periods of the neurons. Phase resetting in the absence of threshold events, such as action potentials or bursts, is completely defined by the motion along the limit cycle. However, in a neurobiological context, the portion of the limit cycle corresponding to the threshold event (a spike or burst) is special [8]. Here we will refer to the event as a burst. The synaptic projections to the target neuron(s) are only active during a burst, therefore the effect on the timing of each burst is critical, not just the total net phase resetting. When a perturbation is received near the end of a cycle (see Fig. 1), it may exert its primary effect not on the cycle in which it is initiated, but on the next cycle, particularly if the perturbation is long. This insight motivated our separate consideration of $F_1(\phi)$, the effect on the first burst, and $F_2(\phi)$, the effect on the second burst.

Our stability analyses require the assumption of a particular firing pattern, and such a stability analysis only holds for that particular assumed firing order. Synchrony poses a particular problem for our method of linearizing about a stable firing pattern: slight deviations from synchrony can change the firing order, invalidating stability results obtained by assuming a particular firing order. Also, the problem with inputs of long duration overlapping into the second burst cycle following a perturbation is exacerbated near synchrony, because the bulk of one of the two perturbations will fall in the second burst after it is perturbed. This work addresses rigorously the effect on the second cycle length after the perturbation.

Historically, several assumptions have been made in order to reduce the complexity of the pulsatile coupling problem to manageable proportions. Winfree [20] simplified the problem by assuming that the individual oscillators are strongly attracted to their limit cycles, so that variations in amplitude can be neglected, and only variations in phase are considered. Phase resetting theory [7, 20] assumes that a perturbation instantaneously displaces the trajectory along the limit cycle, with a phase advance causing the trajectory to skip ahead on the limit cycle and a delay causing it to jump backwards. This has the effect of either shortening or lengthening the particular cycle in which the perturbation occurs. The alteration in the duration of the cycle can be tabulated as the PRC, a function $F(\phi)$ of the temporal phase (ϕ) at which the perturbation is received.

The two criteria for the existence of a periodic pattern or mode are formulated entirely in terms of the open loop PRCs generated as shown in Fig. 1, in which the changes in the perturbed periods $F_1(\phi)$ and $F_2(\phi)$, are tabulated as a function of the phase (ϕ) at which the stimulus is received. The duration of the cycle in which perturbation was initiated is $P_1(\phi)$, since it is a function of the phase of the stimulus. The normalized PRC that quantifies the influence on the first burst is represented by $F_1(\phi)$, where $F_1(\phi) = (P_1(\phi) - P_0)/P_0$, so that a positive value represents a phase delay and a negative value a phase advance. For instance, the duration of the next cycle is $P_2(\phi)$, and the corresponding normalized PRC that quantifies the influence on the second burst is $F_2(\phi) = (P_2(\phi) - P_0)/P_0$. Assuming 1:1 entrainment at a common frequency, these criteria can be stated verbally as (1) the sum of the firing intervals (t_f^i) and/or recovery intervals (t_r^i) between oscillators around any closed loop must add up to the common entrained period, or a multiple thereof, and (2) each presynaptic oscillator must produce the correct amount of phase advance or phase delay in its postsynaptic oscillator in order to arrive at common entrained period. The terminology used in the mathematical formulation is illustrated in Fig. 1, in which a burst is considered to begin when the threshold indicated by the dotted horizontal line is exceeded, and the stimulus used to perturb the neuron is a free-running burst in an identical neuron. The entrained period $P^i[n]$ is determined by the combined phase resetting, such that $P^i[n] = P_0^i(1 + F_1^i(\phi^i[n]) + F_2^i(\phi^i[n-1]))$, where P_0 is the intrinsic period and the index n refers to appropriate cycle of the indexed oscillator as shown on Figs. 2, 4. The entrained firing interval, t_f^i, is equal to the time elapsed between the firing of the reference neuron and that of its preynaptic neuron, and can be mathematically described as $t_f^i = P_0^i(\phi^i[n] + F_2^i(\phi^i[n-1]))$, where the subscript i refers to the reference neuron, and ϕ^i is the phase of this postsynaptic neuron at the time at it begibs to receive a pulse perturbation. The entrained recovery interval t_r^i is the time elapsed between the firing of the presynaptic neuron and the next firing of the reference neuron. This interval can be characterized as $t_r^i = P_0^i(1 - \phi^i[n] + F_1^i(\phi^i[n]))$. The first criterion for existence can then be restated mathematically as $\sum_{k=1}^{N} P_0^k(\phi^{k*} + F_2(\phi^{k*})) = jP_0^n(1 + F(\phi^{n*}))$, where N is the number of oscillators and $F = F_1 + F_2$. The quantity on the left-hand side is the sum of the firing intervals including the correction for $F_2(\phi^*)$, and the quantity on the right-hand side is an integer (j) multiple of the entrained period. The second criterion can be restated mathematically as $P_0^1(1 + F(\phi^{1*})) = P_0^2(1 + F(\phi^{2*})) = ... = P_0^N(1 + F(\phi^{N*}))$. These criteria are the result of a systematic tabulation of periodicity constraints.

2. Results

2.1. Two-neuron Ring

First we will rigorously treat the effects of $F_2(\phi)$ on stability for an example using an alternating mode in a pair of oscillators, then we will show that an approximation will generally suffice in the case of synchrony, whether the convergence is monotonic or oscillatory. Our work on stability is an extension of the work of Perkel et al. [18]. The essential steps in our stability analysis [3] are (1) assume a stable entrainment, (2) tabulate the algebraic relationships between the cycle periods ($P^i[n]$) and either the firing intervals ($t_f^i[n]$) [2, 3] or the recovery intervals ($t_r^i[n]$) [15] in a system of mutually entrained coupled oscillators (3) linearize the PRC about the entrainment points so that the perturbation in the cycle period resulting from a perturbation in the precise firing times can be calculated using the slope of the PRC (m) at that point, and (4) analyze the coupled linear equations that result from 2 and 3 to describe the response to such a perturbation using discrete linear systems theory to determine if such a perturbation grows (unstable) or shrinks (stable) during subsequent cycles. The assumption of a stable entrainment or firing pattern is critical, because the stability analysis only holds for that particular assumed firing order.

Dror et al. [3] considered only the phase resetting of the cycle containing the start of the perturbation and neglected the phase resetting that occurs in the next cycle, which in the case of a long pulse contains the remainder of the pulse (see Fig. 1). We generalized the stability results given in Dror et al. [3] for two reciprocally coupled neurons firing in an alternating order (see Fig. 2A). Figure 2B illustrates the first order mapping logic that was utilized in [3]. The approach in this paper, which considers not only $F_1(\phi)$ but also $F_2(\phi)$, increases the order of the problem, and cannot easily be depicted graphically.

These results can also be generalized to apply tó synchrony. We have formulated these criteria in terms of the slopes (m) of the relevant PRCs at the point of entrainment, using subscripts to denote the PRC with respect to the first (1) or second (2) burst after a perturbation begins, and superscripts to denote which neuron of the heterogeneous pair (1 or 2). The periodicity constrains can be expressed using above defined recovery intervals:

$$\begin{aligned} t_r^1[n] + t_r^2[n-1] &= P^1[n], \\ t_r^1[n] + t_r^2[n] &= P^2[n]. \end{aligned} \tag{1}$$

The analyzes can also develop using the recovery intervals t_r^i to formulate the periodicity constraints.

$$\begin{aligned} P_0^1(1-\phi^1[n] + F_1^1(\phi^1[n])) + P_0^2(1-\phi^2[n-1] + F_1^2(\phi^2[n-1])) \\ = P_0^1(1 + F_1^1(\phi^1[n]) + F_2^1(\phi^1[n-1])), \\ P_0^1(1-\phi^1[n] + F_1^1(\phi^1[n])) + P_0^2(1-\phi^2[n] + F_1^2(\phi^2[n])) \\ = P_0^2(1 + F_1^2(\phi^2[n]) + F_2^2(\phi^2[n-1])). \end{aligned}$$

If ϕ^{k*} is the steady value corresponding to $\phi^k[n]$, and $\Delta\phi^i[n]$ is its infinitesimal

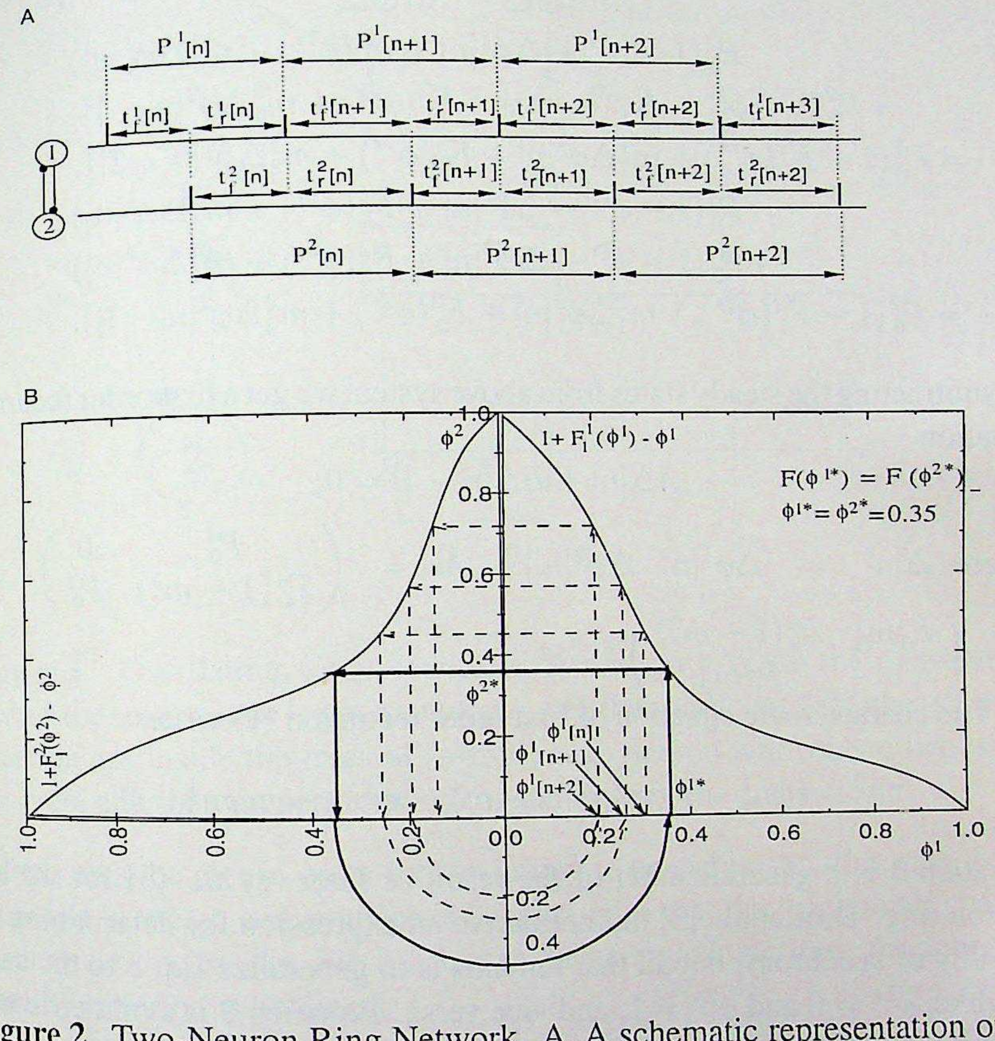
A
P¹[n]
P¹[n+1]
P¹[n+2]
t¹f[n]
t¹r[n]
t¹f[n+1]
t¹r[n+1]
t¹f[n+2]
t¹r[n+2]
t¹f[n+3]
1
2
t²f[n]
t²r[n]
t²f[n+1]
t²r[n+1]
t²f[n+2]
t²r[n+2]
P²[n]
P²[n+1]
P²[n+2]
B
1.0
φ²
1+ F¹1(φ¹) - φ¹
F(φ ¹*) = F (φ²*)
φ¹*= φ²*=0.35
0.8
0.6
0.4
φ²*
0.2
φ¹[n]
φ¹[n+1]
φ¹[n+2]
φ¹*
1+F²1(φ²) - φ²
1.0
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1.0
φ¹
0.2
0.4

Figure 2. Two-Neuron Ring Network. A. A schematic representation of the recurrent inhibition of the postsynaptic neuron in a ring of two coupled neurons. Solid vertical line segments indicates the firing times of the neuron to their left. The recovery interval for the postynaptic neuron (the cophase) is the same with the firing interval of the presynaptic neuron. B. Recovery interval plot against the phase leads to the graphic method of steady modes construction. The closed path indicates a steady (unstable - in this case) mode. The steady mode $\phi^{1*} = \phi^{2*} = 0.35$ is unstable because any small perturbation from this steady state drive the system away (into stable steady state $\phi^{1*} = \phi^{2*} = 0.0$). Indeed, the n-th input received by the first neuron, $\phi^1[n]$, determines its recovery interval (cophase) $1 + F_1^1(\phi^1[n]) - \phi^1[n]$ which is, in the same time, the input phase (firing time) for the second oscillator $\phi^2[n]$ and so on. Every new iteration drives the next input away form $\phi^{1*} = \phi^{2*} = 0.35$ steady state.

perturbation, then the above equations can be rewritten as:

$$\begin{aligned}
&P_0^1(1-\phi^{1*}-\Delta\phi^1[n]+F_1^1(\phi^{1*})+m_1^1\Delta\phi^1[n])\\
&+P_0^2(1-\phi^{2*}-\Delta\phi^2[n-1]+F_1^2(\phi^{2*})+m_1^2\Delta\phi^2[n-1])\\
&=P_0^1(1+F_1^1(\phi^{1*})+m_1^1\Delta\phi^1[n]+F_2^1(\phi^{1*})+m_2^1\Delta\phi^1[n-1]),\\
&P_0^1(1-\phi^{1*}-\Delta\phi^1[n]+F_1^1(\phi^{1*})+m_1^1\Delta\phi^1[n])\\
&+P_0^2(1-\phi^{2*}-\Delta\phi^2[n]+F_1^2(\phi^{2*})+m_1^2\Delta\phi^2[n])\\
&=P_0^2(1+F_1^2(\phi^{2*})+m_1^2\Delta\phi^2[n]+F_2^2(\phi^{2*})+m_2^2\Delta\phi^2[n-1]).
\end{aligned}$$

By subtracting the steady states from above system we get a first order recurrent equation

$$A\Delta[n]+B\Delta[n-1]=0, \tag{2}$$

where $\Delta[n] = [\Delta\phi^1[n] \;\; \Delta\phi^2[n]]^T$, $A = \begin{pmatrix} P_0^1 & 0 \\ P_0^1(1-m_1^1) & P_0^2 \end{pmatrix}$, and $B = \begin{pmatrix} P_0^1 m_2^1 & P_0^2(1-m_1^2) \\ 0 & P_0^2 m_2^2 \end{pmatrix}$.

The characteristic equation of first order recursion (2) writes:

$$\lambda^2-((m_1^1-1)(m_1^2-1)-m_2^1-m_2^2)\lambda+m_2^1m_2^2=0. \tag{3}$$

Equation 3 is a generalization of the result of Dror et al. [3] for the two neuron ring. Dror et al. [3] did not derive an expression for determining the stability of synchrony, but all that remains is to generalize Eq. 3 to the cases in which $\phi^{1*}=0$ and $\phi^{2*}=1$, and vice versa. Equation 3 is symmetric with respect to the two oscillators, so it applies to perturbations from synchrony that cause oscillator 1 to lead as well as those that cause oscillator 2 to lead, if we consider only the possibility of monotonic and not oscillatory convergence. If the contributions of higher order PRCs such as $F_3(\phi)$ are truly negligible, then

$$\lim_{\phi\to 1^-} F_1(\phi)=0, \quad \lim_{\phi\to 0^+} F_2(\phi)=0, \quad \lim_{\phi\to 0^+} F_1(\phi)=\lim_{\phi\to 1^-} F_2(\phi).$$

Under these assumptions, $F_1(\phi)$ is continuous from the left at $\phi=1$ and $F_2(\phi)$ is continuous from the right at $\phi=0$, and so the m_1^i term also approaches zero as ϕ^i approaches 1, and the m_2^i term approaches 0 as ϕ^i approaches 1. If we assume that oscillator 1 leads the ϕ^1 approaches 0 and ϕ^2 approaches 1, so we can ignore m_1^1 and m_2^2, so for stability $0<(m_1^2(\phi)+m_2^1(\phi)-1)<1$. Assuming a perturbation in the opposite direction $0<(m_1^1(\phi)+m_2^2(\phi)-1)<1$. Both

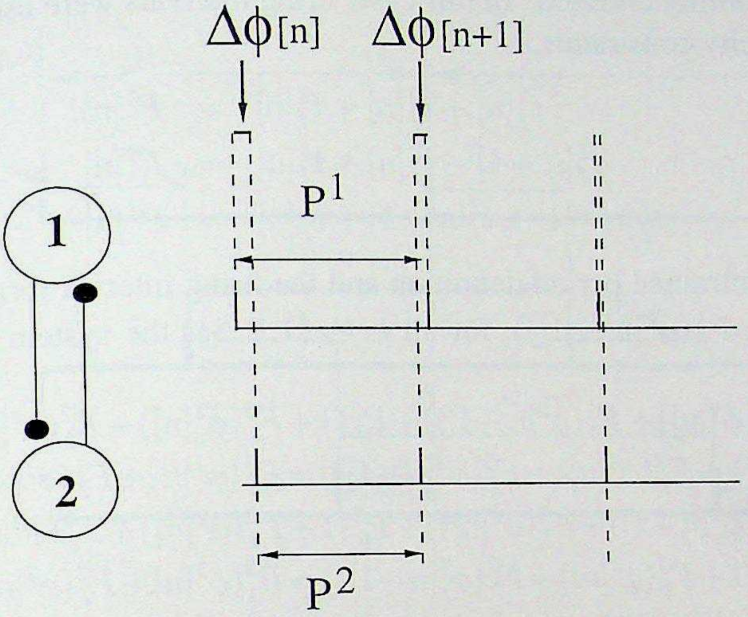

Figure 3. Oscillating Convergence to Synchrony. When the convergence toward the steady state is not monotonic, i.e., the solutions of the characteristic equation are inside the interval $(-1, 0)$, each neuron will receive two inputs per cycle and then no perturbation for subsequent one and so forth.

criteria must be met in order for synchrony to be stable. Another case can be proposed in which convergence is oscillatory in that the firing order changes in alternate cycles (see Fig 3). Using the same assumptions, the following stability criterion is obtained:

$$-1 < (m_1^1(\phi) + m_2^2(\phi) - 1)(m_1^2(\phi) + m_2^1(\phi) - 1) < 0.$$

For pairs of identical oscillators, oscillating convergence to stability cannot be achieved because the eigenvalue will be the square of a real number, so it cannot be less than zero.

2.2. Three-neuron Ring

As an example of how the method above can be extended to rings containing more than two neurons, an example is worked out for one particular firing order in a three-neuron ring.

We use as a representative example the particular firing order depicted in Fig. 4A. Different firing patterns, such as that shown in Fig. 4B, produce a different stability criterion. In this case, firing intervals were used to tabulate the periodicity constraints.

$$\begin{aligned} t_f^1[n] + t_f^2[n] + t_f^3[n] &= P^1[n], \\ t_f^1[n+1] + t_f^2[n] + t_f^3[n] &= P^2[n], \\ t_f^1[n+1] + t_f^2[n+1] + t_f^3[n] &= P^3[n]. \end{aligned} \tag{4}$$

Using the entrained period definition and the firing interval formula $t_f^k[n] = P_0^k(\phi^k[n] + F_2^k(\phi^k[n-1]))$, for all $k \in \{1,2,3\}$, the system in (4) above becomes

$$\begin{aligned} &P_0^1(1+F_1^1(\phi^1[n])+F_2^1(\phi^1[n-1]))+P_0^2(1+F_2^2(\phi^2[n])+F_2^2(\phi^2[n-1])) \\ &\qquad +P_0^3(1+F_1^3(\phi^3[n])+F_2^3(\phi^3[n-1]))=P_0^1(\phi^1[n]+F_2^1(\phi^1[n-1])), \\ &P_0^1(1+F_1^1(\phi^1[n+1])+F_2^1(\phi^1[n]))+P_0^2(1+F_2^2(\phi^2[n])+F_2^2(\phi^2[n-1])) \\ &\qquad +P_0^3(1+F_1^3(\phi^3[n])+F_2^3(\phi^3[n-1]))=P_0^2(\phi^2[n]+F_2^2(\phi^2[n-1])), \\ &P_0^1(1+F_1^1(\phi^1[n+1])+F_2^1(\phi^1[n]))+P_0^2(1+F_2^2(\phi^2[n+1])+F_2^2(\phi^2[n])) \\ &\qquad +P_0^3(1+F_1^3(\phi^3[n])+F_2^3(\phi^3[n-1]))=P_0^3(\phi^3[n]+F_2^3(\phi^3[n-1])). \end{aligned}$$

The corresponding equations for the infinitesimal perturbations $\Delta\phi^i[n]$ are:

$$\begin{aligned} &P_0^1(1-m_1^1)\Delta\phi^1[n] + P_0^2(\Delta\phi^2[n] + m_2^2\Delta\phi^2[n-1]) \\ &\qquad +P_0^3(\Delta\phi^3[n] + m_2^3\Delta\phi^3[n-1]) = 0, \\ &P_0^1(\Delta\phi^1[n+1] + m_2^1\Delta\phi^1[n]) + P_0^2(1-m_1^2)\Delta\phi^2[n] \\ &\qquad +P_0^3(\Delta\phi^3[n] + m_2^3\Delta\phi^3[n-1]) = 0, \\ &P_0^1(\Delta\phi^1[n+1] + m_2^1\Delta\phi^1[n]) + P_0^2(\Delta\phi^2[n+1] \\ &\qquad +m_2^2\Delta\phi^2[n]) + P_0^3(1-m_1^3)\Delta\phi^3[n] = 0. \end{aligned}$$

The above equations can be written as a second order matrix recursion:

$$A\Delta[n+1] + B\Delta[n] + C\Delta[n-1] = 0, \tag{5}$$

where $\Delta[n] = [\Delta\phi^1[n] \;\; \Delta\phi^2[n] \;\; \Delta\phi^3[n]]^T$ and

$$\begin{aligned} A &= \begin{pmatrix} P_0^1(1-m_1^1) & P_0^2 & P_0^3 \\ P_0^1 & 0 & 0 \\ P_0^1 & P_0^2 & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & P_0^2 m_2^2 & P_0^3 m_2^3 \\ P_0^1 m_2^1 & P_0^2(1-m_1^2) & P_0^3 \\ P_0^1 m_2^1 & P_0^2 m_2^2 & P_0^3(1-m_1^3) \end{pmatrix}, *C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P_0^3 m_2^3 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

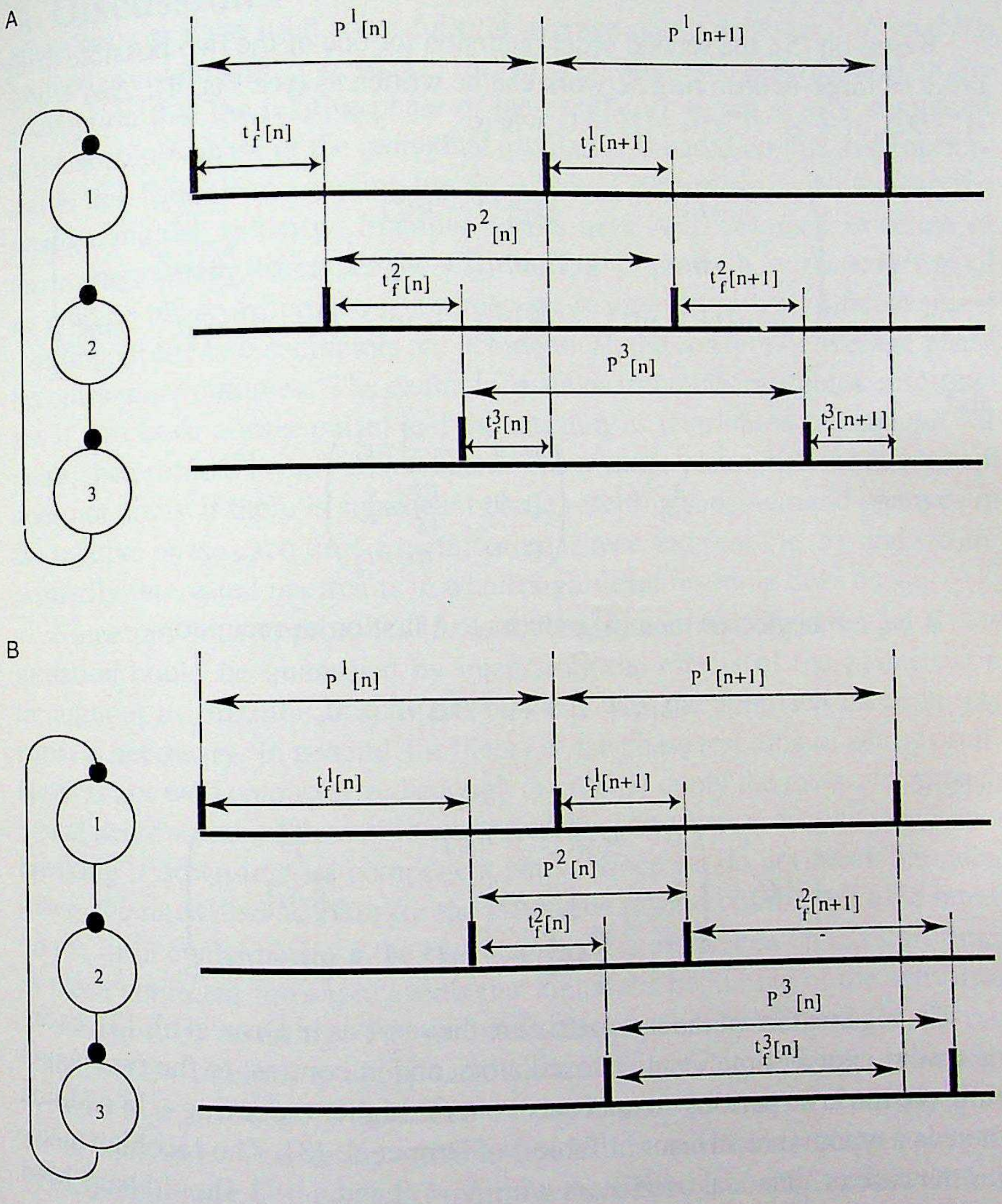
A
p¹[n]
p¹[n+1]
t_f¹[n]
t_f¹[n+1]
p²[n]
t_f²[n]
t_f²[n+1]
p³[n]
t_f³[n]
t_f³[n+1]
1
2
3
B
p¹[n]
p¹[n+1]
t_f¹[n]
t_f¹[n+1]
p²[n]
t_f²[n]
t_f²[n+1]
p³[n]
t_f³[n]
1
2
3

Figure 4. Three-Neuron Ring Network. A schematic representation of the recurrent inhibition of the postsynaptic neuron in a ring of three coupled neurons. Solid vertical line segments indicates the firing times of the neuron to their left. There are two distinct steady-state modes that differ in the firing order of the neurons. The mod (A) has $j = 1$ meaning that the sum of the phase differences between successive oscillators is one entrained period. The mode (B) has $j = 2$ and the sum of the phase differences between successive oscillators is twice of the entrained period.

Based on (5), the second order recursion for one of the two possible firing order of three-neuron ring network can be written as (see Fig. 4): $\Delta[n+1] + A^{-1}B\Delta[n] + A^{-1}Cx[n-1] = 0$, where

$$A^{-1}B = \begin{pmatrix} -m_2^1 & -\frac{P_0^2}{P_0^1}(1-m_1^2) & -\frac{P_0^3}{P_0^1} \\ 0 & 1-m_1^2-m_2^2 & \frac{P_0^3}{P_0^2}m_1^3 \\ \frac{P_0^1}{P_0^3}m_2^1(1-m_1^1) & -\frac{P_0^2}{P_0^3}m_1^1(1-m_1^2) & 1-m_1^1-m_1^3-m_2^3 \end{pmatrix},$$

and

$$A^{-1}C = \begin{pmatrix} 0 & 0 & -\frac{P_0^3}{P_0^1}m_2^3 \\ 0 & 0 & \frac{P_0^3}{P_0^2}m_2^3 \\ 0 & 0 & -m_1^1 m_2^3 \end{pmatrix}.$$

If m_2^k are neglected then (5) reduces to a first order recursion

$$\Delta[n+1] + A^{-1}B\Delta[n] = 0,$$

where

$$A^{-1}B = \begin{pmatrix} 0 & -\frac{P_0^2}{P_0^1}(1-m_1^2) & -\frac{P_0^3}{P_0^1} \\ 0 & 1-m_1^2 & \frac{P_0^3}{P_0^2}m_1^3 \\ 0 & -\frac{P_0^2}{P_0^3}m_1^1(1-m_1^2) & 1-m_1^1-m_1^3 \end{pmatrix}.$$

The eigenvalues of the last matrix are the same as in Dror et al. [3], except that we do not assume identical oscillators, and in contrast to the two-neuron ring, the intrinsic periods do not cancel out for higher order rings. In addition, there is a typographical error in Table 1 of Dror et al. [3]. The Jacobian matrix for the case of identical oscillators with $N = 3$ and $j = 1$ should have been, using our notation rather than that of Dror et al. [3]:

$$\begin{pmatrix} 1-m_1^1 & m_1^1 \\ -m_1^3(m_1^1-1) & 1-m_1^2-m_1^3 \end{pmatrix}.$$

A similar approach can be applied for the firing order "2 → 1 → 3" (see Fig. 4B).

3. Discussion

Many authors have used some form of average phase difference to analyze coupled oscillators [5, 6, 4, 10, 11, 13, 12]. This approach is predicated on the assumption that the relative phase of the oscillators varies slowly compared to the absolute phase of the individual oscillators is based on this assumption, and in this formalism coupling has the effect of increasing or decreasing the average angular velocity. Stability results have been obtained in terms of the average phase difference [9, 17], but Fig. 5 shows how the concept of an average phase difference is not relevant to circuits with significant phase resetting. If the two oscillators are not identical, the concept of average phase becomes more complex. The example assumes instantaneous phase resetting, but if you have a long pulse, and the resetting is distributed throughout the pulse, the problem is only exacerbated. The average phase difference concept does not apply if there is significant phase resetting causing rapid changes in the relative phase each time a perturbation is received (see Fig. 5), and we are primarily interested in circuits in which significant resetting does occur.

Since previous approaches [5, 6, 4, 9] assumed that the effect of a perturbation could be quantified by integrating the effects of the perturbation throughout its duration, it may not be clear why the approach taken in this paper is necessary. In general, the theory of the phase resetting of neural oscillators is not well characterized enough in order to apply the integral approach to real neurons, hence our approach to treat the perturbation as a whole without breaking it down into its component parts. Since we do not break the pulse down, we must also account for the privileged position of the spike (or burst) on the limit cycle, because the entire analysis is predicated on the start times of these nonlinear threshold events that signal the beginning of the activation of the pulsatile coupling. The results in this paper were derived for discrete maps, but the intent is to apply these methods to neural oscillators, which are systems that are actually described by continuous systems of coupled differential equations. The weakest assumption in this analysis is that the trajectory of each coupled oscillator must return to the limit cycle after each activation of the pulsatile coupling before the next activation. If this assumption is not valid, then higher order terms such as $F_3(\phi)$ make a contribution to stability, but are not accounted for in our stability analysis. However, the results are more general than those previously published by Dror et al. [3]. Generalization of the approximation to synchrony in higher order rings and in less restrictive architectures such as all-to-all networks has not yet been successfully completed. However, our experience with the emulation of ring circuits

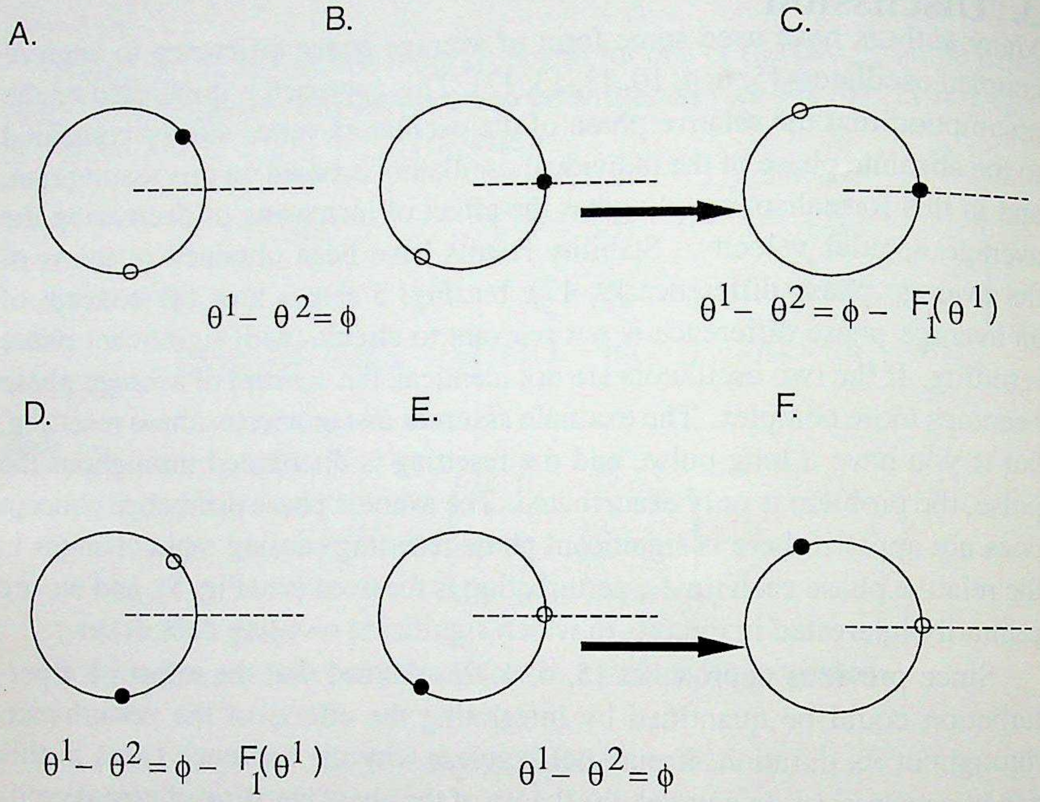

Figure 5. Phase Differences Are Not Constant When There is Resetting. The large circle in the panels above represents the limit cycle associated with each of two identical, identically coupled oscillators. The small open and filled circles indicate the respective positions of the two oscillators along their respective limit cycles at a given instant in time. The coupling is pulsatile, and the pulse perturbation of one oscillator is activated when the other oscillator reaches threshold, which is indicated by the dashed line. The absolute phase of each oscillator is given by θ^i, whereas the relative phase is given by their difference (see [1, 2]). At threshold, the absolute phase θ^i is defined as zero. The example shown is symmetric, in that each oscillator receives an input (i.e., its partner reaches threshold) when its absolute phase is equal to ϕ. Thus in A, the relative phase is ϕ, as it is in B, then the oscillator indicated by the filled circle reaches threshold. The phase of the oscillator indicated by the open circle is reset by $F_1(\phi)$, so that in C, the relative phase is now $\phi - F_1(\phi)$. This is also true in D, until the oscillator represented by the open circle reaches threshold and resets the oscillator represented by the filled circle, so that once again in F the relative phase is ϕ. The contribution of $F_2(\phi)$ was ignored in these simplified drawings.

of oscillations described by systems of nonlinear differential equations using discrete maps based on phase resetting indicates that at least in rings of order 3 and above the contribution of the m_2^i terms are important. Synchronous modes in rings of order three and four were observed to lose stability in numerical simulations of the discrete mapping when the contributions of the m_2^i terms were ignored. We hypothesize that this is because multiplicative terms involving the m_2^i terms first appear in the matrix recursion for rings in which $N = 3$, whereas only additive terms appear for $N = 2$.

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Nonlinear Dynamics and Chaos in Information Processing Neural Networks

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Abstract

We consider a number of possible roles of complex dynamics and chaos in information processing by neural networks. First, we review the working principles of some well-known neural networks, and then discuss a number of approaches to utilization of chaos in neural networks. Our main goal is to present a novel view of the problem of chaos in information processing. We demonstrate that chaos emerges naturally in controls when a neural network forms a controlling part of a more complex system. We show that such neural networks can enhance efficiency by using chaos for explorations in a method known as Reinforcement Learning. A discussion on Hamiltonian neural networks is also included.

1. Introduction

Artificial neural networks (ANNs) are widely used now in many engineering applications and research problems [17, 18, 45, 46, 59, 58, 79, 95, 97]. Since ANNs are efficient tools for information processing, there is an ongoing quest for improving their performance, widening the areas of applications and finding new working principles. About ten tears ago, attempts were made to enhance the performance of ANNs on the basis of their complex or chaotic temporal

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behaviors (see e.g., [93, 34] for the review). The question is: can chaos be useful for information processing? There are arguments both in favor and against the question of usefulness of chaos. On one hand, activities of the brain demonstrate complex and possibly chaotic temporal behaviors, that suggests that maybe the brain uses chaos for sustaining life. On the other hand, neural networks that are currently in use for practical purposes are designed to be nonchaotic on the presumption that chaos is not needed for information processing (though chaotic information processing systems exist, e.g., [5]).

The purpose of this paper is to reflect on the underlying principles of operation of existing neural networks from the point view of nonlinear dynamics, to discuss a number of attempts to endow networks with chaotic behavior, and to present a new perspective of the role of chaos in neural networks. If we consider a neural network as an element of a larger system interacting with the world, then dynamical chaos can emerge in rather simple models. A number of such models are known, for example, in artificial intelligence. Moreover, systems interacting with their surroundings need a source of 'initiatives' to pursue exploration and learning from experience. Dynamical chaos can serve as a source of such initiatives. In this work, we also discuss Hamiltonian neural networks that have received little attention so far in information processing. Hamiltonian neural networks have the advantage that their quantum analogs can be studied.

The paper is organized as follows. First, in Section 2, we discuss definitions and various viewpoints of neural networks. In Section 3, we consider working principles of some existing neural networks to explain why, in the opinions of users of these networks, complex dynamics and chaos are not necessary. In Section 4, we give reasons why chaos may be useful. In Sections 5, 6, and 7 we discuss different approaches to the problem of chaos and complex behavior in neural networks.

2. What is an Artificial Neural Network?

In spite of the growing number of publications in the field, there is no consensus on the precise definition of ANNs. The reason for this is that there are too many types of them. Sometimes a more general term called "connectionism" is used for ANNs. The term connectionism means a methodology of making a complex system by a combination of connected elements that are similar or identical [29]. The basic nonlinear elements of an ANN are called formal neurons. Typically, a formal neuron receives a number of input signals x_i, sums them up, performs a nonlinear transformation of this sum $y = f(\sum x_i)$ and sends y as its output to other neurons or devices. The function f is called the "activation function." Quite often, f is a "sigmoid" function of the form $f(x) = 1/(1 + e^{-ax})$ or $f(x) = \tanh ax$. In the limiting case $a \to \infty$, f becomes a threshold function.

The connections among neurons are characterized by the weights w_{ij}. Some or all of the neurons may receive external input signals X_k with weights w'_{ik}. Therefore, a typical operation performed by a neuron in the network is a transformation $f(\sum_j w_{ij}x_j + \sum_k w'_{ik}X_k)$. After all the necessary internal computations are carried out, the output of the network consists of the states of some or all of the neurons. Therefore, an ANN has the following general features:

- It is an information processing tool that receives an input signal X (usually a multicomponent vector) and generates the output $Y = F(X)$ (Y may be a vector).
- It consists of a number of similar neurons that usually operate in parallel like the neurons of the real brain.
- It is capable of learning which is normally understood as adjusting the parameters w_{ij} to achieve certain desired properties of the mapping F.

ANNs with these general features are studied from different viewpoints of which the following four are the most important ones.

2.1. The Viewpoint of Statistics and Data Analysis

One of the problems that often arise in practice involves approximation of unknown functions from experimentally observed data points. That is, given the set of pairs of points $\{X_i, Y_i = \Phi(X_i)\}$, it is required to find an approximation $F(X)$ of $\Phi(X)$. Besides usual curve fitting, an astonishingly large number of problems such as pattern recognition, classification, prediction, and control can be reduced to this form. For example, X_i may be the set of sensor data, representing situations around a car, and Y_i may be the actions of an experienced human operator for safe driving. Good approximation $F(X)$ gives a control system the ability to drive a car.

Classical approximation theory is very well developed. However, it usually deals with one dimensional functions. When we need to approximate an unknown function of, say ten variables, classical methods such as polynomial approximations, are of little help. Approximations by compositions of sigmoid functions are more effective.

So, the general framework of the approach is as follows. We choose a certain form of the approximating function $F(X, w)$, and adjust w to minimize the error $\sum |Y_i - F(X, w)|^2$. The process of minimization will be the learning of the network. The process of learning essentially depends on the specific implementation of the network.

At the first glance, this seems very simple. But if one tries to build a network, one immediately faces a number of questions regarding the number of neurons to be taken, organizations of the connections to be followed, type of minimization to be used, and so on. The modern theory of neural networks can answer some of them on a rigorous basis, and others on the basis of a vast experience. The rigorous part is based on (a) Kolmogorov theorem on functions approximation by sigmoid-type functions, (b) a number of general approximation theorems which state that multilayer perceptrons (see below) are universal approximators, and (c) the methods of statistics. There is a very good description of statistical view of neural networks, e.g., in [16, 25, 81].

2.2. Computer Science Viewpoint

This direction is related mainly with logic, circuitry and computation capabilities of neural networks. The mapping $X \to Y$, provided by a neural network, can be considered as a way of performing computation. In this computational aspect, the main differences between neural networks and usual computers are that (1) ANNs can work in a highly parallel way, (2) instead of programming, ANNs can learn from examples, and (3) ANNs can recognize patterns even when only partial informations about the patterns are available.

Nonetheless, there is no gap in the history of neural networks and computers. The beginning of the history of ANNs often refers to the famous 1943 paper of McCulloch and Pitts [63]. This paper showed that simple networks of threshold elements (sometimes they are called McCulloch-Pitts neurons) can work as logical gates, and hence they can form a large network that can work as a universal computer.

Another interesting aspect related mainly with artificial intelligence is learning of a different type. For data analysis, if the output for a given input is known, the difference between the actual and desired outputs can be used to update the weights (supervised learning). In machine learning, for example in the problem of robot navigation, it is necessary to learn how to take the correct action at the current 'state' (the set of sensory data) when the answer is unknown. Only after a long sequence of trials and errors, it is possible to learn about the appropriate actions that should be taken. Such a learning is called reinforcement learning and it is based on reinforcement of reward (for success) and punishment (for failure). The first attempt to build a reinforcement learning neural network was made as early as 1954, but significant progress in developing the method of reinforcement learning was achieved during 1980-1990 [83]. An important feature of this type of learning involves integration of a neural network into a larger system, and the learning proceeds as an interaction of a learning agent with its environment. We shall return to this matter later in Sections 5 and 6.

An excellent account of the computer science viewpoint of neural networks is presented, e.g., in [7].

2.3. The Biological Viewpoint

Considerations of computing neural circuitry and limitations of the capabilities of computing devices inevitably send us back to the beginning of the theory of neural networks, that is, to biological analogies, cognitive science and theories of the brain (e.g., [78]). Or in other words, we have the biological viewpoint of neural networks. An excellent description of this viewpoint can be found, for example, in Freeman's review [34, 33].

Studies of the brain in psychology and physiology have a much longer history than that of neural networks. For biology, the technique of artificial neural networks is only one of many possible approaches. The main task of neural networks in the biological framework is to model and explain the results of physiological experiments, or to deepen our understanding of how the brain works. Moreover, such modeling helps test biological hypotheses that can not be verified any other way [34].

The whole field of neural network studies has been inspired by discoveries in neuroscience. The hope of modeling the principal brain functions was very high in 1950s. This was so because originally the complexity of the brain was underestimated. The first estimates gave the number of neurons as $\sim 10^3$ [34]. Later it was estimated that the brain contains about 10^9 neurons, and each of them might have up to 10^5 connections. So, the task of modeling the functions of the whole brain has been postponed to indefinite future.

The work on modeling biological structures began with a single neuron or its parts. It should be noted that a biological neuron is much more complex than a formal neuron of an ANN. For example, the model of electric signal propagation along axon ("output" structure of a neuron) includes two partial differential equations while a simple description of the neuron dynamics requires delay-differential equations (see e.g., references in [34]). On the next level of description, there were models of groups of neurons or small brain structures [34]. In this case the main task is to construct a network which would reproduce the features observed in real physiological experiments. Such models may be totally different compared to networks with approximation purposes. Also they are substantially more complex.

Important aspects of mathematical biology of neural networks include classification of various models, search for the most general model, and establishing the principles of their construction. A significant progress in this field has been made with the help of methods of bifurcation theory [53].

Numerous physiological experiments demonstrate the complexity of the behavior of the brain: most probably, it is chaotic [34, 8, 9, 38, 41]. But why? What are the advantages of chaos in information processing? Should data processing systems possess their internal dynamics and what would be the principles of their work? Such questions are akin to nonlinear dynamics.

2.4. The Viewpoint of Nonlinear Dynamics

From the framework of nonlinear dynamics, the important issues are the general properties of system's dynamics. Nonlinear dynamics may be of little help for many specific details, but it does provide a basis for general view of various neural networks.

This point of view helps to classify all neural networks into two big categories: "functions" and "dynamical systems." Our main interest is in the latter category, although we will say a few words about the former one. Dynamical systems can be conservative or dissipative. All the dynamical neural networks that we know of are dissipative. Dissipation plays an important role in pattern recognition. It eliminates noisy or irrelevant features from the input data and transforms a distorted pattern into a "correct" one. That is, the process of pattern recognition is considered as convergence to attractors of dynamical systems. Conservative dynamical systems do not have attractors and, therefore, they can not implement this scheme of recognition. Nonetheless, in Sect. 7 we will consider an example of a conservative Hamiltonian dynamical system for pattern recognition. Such systems are of interest from the point of view of quantum information processing.

Attractors of dissipative dynamical systems represent the output (Y) of a neural network. The input X can be fed into it in two major ways: as initial data or as parameters of the dynamical system. Combinations of these ways are possible, but we do not know of any such networks. Therefore, nonlinear dynamics provides the following classification of all existing neural networks according to the method of building the mapping $X \to Y$.

1. Direct functional representation. The resulting output is obtained by a number of explicit transformations of the input. The stages of transformation involve the neurons and weights. Often such neural networks are simply multidimensional functions depending on parameters: $Y = F(X, w)$, and the functional form of F is known (feed-forward nets). In principle, there can be loops in the function definition. For example, a part of the output can be fed back as input to generate an implicit function such as $Y = F(X, Y, w)$ or some other complex form (recurrent nets). The well known examples of functional networks are (a) multilayer perceptrons and (b) radial basis function networks [16, 45].

2. Dynamical system mapping I: $X =$ initial state, $Y =$ final state (attractor). The dynamical systems may be described by (i) differential equations (DE) in which space and time are continuous, (ii) mappings in which space is continuous and time is discrete or (iii) automata in which space and time are discrete. The final state Y of the mapping $Y = F(X)$ does not depend on X if X is in the basin of attraction of a given attractor. Therefore, this kind of networks may be interesting only when there are multiple attractors. For this reason we shall call this approach as one of multiple-attractor network. To define the mapping F properly, that is to teach the network, one should be able to place the dynamical system's attractors to desired places and to avoid spurious attractors. The best known types are Hopfield networks, brain state in a box, Haken's synergetic computer [49, 45, 27, 42].

3. Dynamical system mapping II: $X =$parameters of dynamical system, $Y =$ its attractor. Again the dynamical system may be of different types,

but in this case the attractor is unique subject to the input. Multiple attractors may spoil the performance. The role of the input is to manipulate this single attractor in a desired way. For this reason, we shall call this approach as one of attractor manipulation [45]. The well-known types of such networks are Grossberg's adaptive resonance maps, Hopfield-Tank combinatorial optimization network, and Freeman's models of olfactory system [18, 99, 51].

4. Hamiltonian neural networks. These networks have not received much attention in the context of information processing by neural networks.

From the point of view of nonlinear dynamics, the most interesting features of ANNs are their dynamical properties like complexity, stability, and predictability. In particular, a very interesting question is, can dynamical chaos improve the performance of neural networks or can it play an essential role in the operations of ANNs? Our attempt to address this question is the main purpose of this article. But, let us first consider in more detail the principles of operation of some important types of networks.

3. Basic Types of ANN

In this section we briefly review several basic types of neural networks from the nonlinear dynamics viewpoint. Our goal is not to present an elaborate review of ANNs, and hence the choice of models and their discussions are limited.

3.1. Functional Networks

Let us consider a simple example of approximating a function $Y = f(X)$ using its values $Y_i = F(X_i)$ at a number of points $X_i \in [X_{01}, X_{02}]$ with the sigmoid-type function $\tanh(x)$. Since hyperbolic tangent has values only in $[-1, 1]$, first we need to make sure that the values of $F(X_i)$ on $[X_{01}, X_{02}]$ also belong to this interval. This can be achieved by a linear transformation (data preprocessing) of Y_i, and we assume that already $Y_i \in [-1, 1]$. Then one has to choose the form of the approximating function $Y = F(X, \mathbf{w})$, where $\mathbf{w}$ is the set of parameters. For example, if the original dependence is close to being linear, one can try $Y = \tanh(w_1 X + w_2)$. Then one has to "teach" this approximation by examples, that is to find the best values of w from the known pairs X_i, Y_i. The parameters can be adjusted by minimizing the error function $E(\mathbf{w}) = \sum_i (Y_i - F(X_i, \mathbf{w}))^2$.

How does it relate to dynamics? The calculation of the result does not contain any dynamics at all. The process of minimization can take the form of a dynamical system, e.g., $\dot{\mathbf{w}} = -\nabla E(\mathbf{w})$, but any complexity in this process is considered as a shortcoming.

Nonetheless, dynamics comes into play as soon as one tries to use implicit functions, for example, $Y = \tanh(w_1 X + w_2 Y + w_3)$. Such implicit representations are more versatile, but they require a procedure to find Y . This is done with the help of the dynamics, e.g., $\dot{Y} = -Y + \tanh(w_1 X + w_2 Y + w_3)$. In this single equation no complex dynamics can exist, but in its multidimensional analogs complex dynamics may be possible. But still it is considered as a serious disadvantage since the main goal — obtaining a single number Y — remains unaccomplished.

Multilayer perceptrons generalize the above example. They have a number of "sigmoid" neurons with states denoted by x_i. Neuron i receives signals from other neurons through the connection weights w_{ij}, and it also receives the input signal X_i from outside the network. The computations performed by the network can be expressed as

$$x_i = f\left(\sum_{j=1}^{N} w_{ij} x_j\right) + X_i. \qquad (1)$$

The function f is sometimes called activation function, and quite often it is a sigmoid function.

A generalization of an explicit function $Y = F(X, w)$ is a class of feedforward networks for which $w_{ij} = 0$ for $j \geq i$. For such a network, the first neuron receives signal only from the outside, the second neuron receives signals from outside and possibly from the first one and so on. For the training of such networks, there is a method of error minimization called error backpropagation. In order to update the weights, one has to calculate the auxiliary error terms e_i for each neuron. It can be shown that e_i satisfy a linear system of equations with upper triangular matrix. Therefore, x_i are calculated from 1 to N, while e_i are calculated from N to 1.

Generalizations of implicit functions $Y = F(X, Y, w)$ are recurrent neural networks. For them there are no restrictions on the matrix w_{ij}. For solving

Eq.(1), one has to apply some iterative scheme, that is, to transform the nonlinear system of equations into a dynamical system. For example, instead of $x = f(wx) + X$, consider the ODE system $\dot{x} = -x + f(wx) + X$ the fixed points of which give solutions of the nonlinear system. The learning scheme involving calculation of the error terms e_i can be generalized for such systems [74, 45], but the equations for e_i do not have such a simple form.

We give here a brief description of the backpropagation learning for the general form of perceptron (1). Let there be the set of M training examples, including the values of input and desired output: $(X^{(k)}, Y^{(k)})$, $k = 1, \dots, M$. Let us also denote by $x_i^{(k)}$ the value of i-th neuron when the input $X^{(k)}$ has been given and the solution of (1) has been obtained. For output neurons we define the error signal $E_i^{(k)} = x_i^{(k)} - Y_i^{(k)}$, for other non-output neurons $E_k = 0$. Then the total approximation error for the given set of parameters w_{ij} is

$$S = \sum_{k=1}^{M} \sum_{p=1}^{N} \left(E_p^{(k)} \right)^2 .$$

The usual way of updating weights is the gradient descent minimization of S, that is

$$\delta w_{\alpha\beta} = -\epsilon \frac{\partial S}{\partial w_{\alpha\beta}} = -\epsilon \sum_{k=1}^{M} \sum_{p=1}^{N} E_p^{(k)} \partial x_{p,\,\alpha\beta}^{(k)}, \tag{2}$$

here we denote $\partial x_{p,\,\alpha\beta}^{(k)} = \frac{\partial x_p^{(k)}}{\partial w_{\alpha\beta}}$ for the sake of brevity. The values of $\partial x_{p,\alpha\beta}^{(k)}$ can be obtained by differentiating (1) with respect to $w_{\alpha\beta}$,

$$\partial x_{i,\,\alpha\beta}^{(k)} = f_i' \cdot \left(\delta_{\alpha i} x_\beta^{(k)} + \sum_{j=1}^{N} w_{ij} \partial x_{j,\alpha\beta}^{(k)} \right),$$

here f' is the derivative of f by its argument. If we denote $L_{ij} = \delta_{ij} - f_i' w_{ij}$, then we can express $\partial x_{j,\,\alpha\beta}^{(k)}$ through the inverse of L:

$$\partial x_{j,\,\alpha\beta}^{(k)} = \sum_{i=1}^{N} L_{ji}^{-1} f_i' \delta_{\alpha i} x_\beta^{(k)} = L_{j\alpha}^{-1} f_\alpha' x_\beta^{(k)} .$$

Then (2) takes the form

$$\delta w_{ij} = -\epsilon \sum_{k=1}^{M} \sum_{p=1}^{N} E_p^{(k)} L_{pi}^{-1} f_i' x_j^{(k)} = -\epsilon \sum_{k=1}^{M} e_i^{(k)} f_i' x_j^{(k)} , \tag{3}$$

where $e_i^{(k)} = \sum_{p=1}^{N} E_p^{(k)} L_{pi}^{-1}$ are the "auxiliary error signals." So, knowing $e^{(k)}$, we can easily calculate weight updates for all weights at once. e should satisfy the system of equations $L^T e^{(k)} = E^{(k)}$, where L^T is the transposed L. For feed forward networks L is lower triangle, L^T is upper triangle, and the system for e can be easily solved.

Often neurons in perceptrons are arranged in layers, and within a layer neurons are not connected to each other. It means that w has a block form with many zero entries. Layering leads to slightly bigger but computationally efficient formulas which can be found in many books. For recurrent perceptrons solving equations for x and e can be combined [74].

Applications of recurrent perceptrons may encounter some difficulties because (a) a solution may not exist, (b) if a solution exists it may not be unique, or (c) the iterative scheme may lead to some type of nontrivial behavior instead of convergence to a fixed point.

Another common type of "functional" network involves radial basis functions. One chooses a scalar basis function $\varphi(r)$ of one variable along with a number of nodes X_{0k}, $k = 1, \ldots, M$. Then one seeks the approximation in the form $F(X) = \sum_{k=1}^{M} A_k \varphi(\|X - X_{0k}\|)$. The values of A_k can be found from minimization of the error function $E(A) = \sum (Y_i - F(X_i))^2$. The solution exists for a large class of basis functions. Also in this case, dynamics is absent.

Nonetheless, the considered types of neural networks are closely related to nonlinear dynamics. They are used for approximating unknown equations of motion of dynamical systems from a time series. The time series gives the pairs of X_k (the current state of the dynamical system) and Y_k (the next state), and the task is to approximate the dependence $Y = F(X)$. Afterwards this approximation can be used for time series prediction [96] (there is even a special type of perceptrons for processing time series and a version of learning called "backpropagation through time") [45].

3.2. Multiple-attractor Networks

3.2.1. A Simple Example

Let us consider a gradient system with a double-well potential,

$$\dot{x} = x - x^3, \qquad x(0) = X.$$

If we take $Y = x(\infty)$ as the output, for $X < 0$ we get $Y = -1$, and for $X > 0$ we have $Y = 1$, so that $Y = \mathrm{sgn}(X)$. This dynamical system recognizes the sign of initial data and can classify them into two clusters. Or we can say that we have stored two "patterns" (+1 and −1, one bit) and the initial data X represent a "noisy" or "distorted" pattern. Then the system recognizes the input data as one of the two stored patterns.

All multiple-attractor networks work in a similar way: (1) stored patters correspond to attractors of the dynamical system; (2) classification or recognition of patterns is done according to the basins of attraction; and (3) convergence to one of the attractors is considered as the process of pattern recognition. This interpretation implies that the number of attractors should be greater than one.

3.2.2. Hopfield-type Networks

These networks are the most well known multiple-attractor networks. The networks are fully connected. The equations of motion have the usual form

$$\hat{x}_i = f\Big(\sum_j w_{ij} x_j + \theta_i\Big), \tag{4}$$

where $\hat{x}_i$ stands for the value of x_i at the next time step. The important feature of the Hopfield [49, 50] model is asynchronous evolution: at every time step only one neuron changes its state while the states of the other neurons remain the same. Sometimes the term "Hopfield network" is applied to a wider class of model with synchronous evolution and partial connections.

In the simplest form the neurons are "binary," i.e., they can only take the values ±1, and f is the sign function $f(u) = \operatorname{sign}(u)$ and $\theta_i = 0$.

Learning of this network is performed by storing the binary patterns $\xi^{(k)}$, $\xi_i^{(k)} = \pm 1$, according to the Hebbian rule,

$$w_{ij} = \frac{1}{N} \sum_{k=1}^{M} \xi_i^{(k)} \xi_j^{(k)}, \qquad i \neq j, \quad w_{ii} = 0, \tag{5}$$

$i, j = 1, \ldots, N$. This means that w_{ij} increases if $\xi_i^{(k)} = \xi_j^{(k)}$, and decreases otherwise.

An interesting feature of this network is the existence of an "energy" functional or Lyapunov function

$$E = -\sum_{i,j=1}^{N} w_{ij} x_i x_j.$$

It is easy to show that at every step E does not increase: if x_i remains the same, so does E. If x_i changes ($\hat{x}_i = -x_i$), then $\Delta E < 0$. Since E is bounded from below, the evolution will stop at an equilibrium point corresponding to local energy minimum.

The main weakness of this type of networks is low storage capacity. If the stored patterns are not specially chosen (orthogonal), usually the system is capable of correctly recognizing about $0.14N$ patterns. If one tries to store more patterns, then the resulting attractors cease to correspond to original patterns. To overcome the problem of false memories and to increase the storage capacity of the network several approaches were proposed, see e.g., [45, 37, 72, 66]. We shall return to this problem later, when we discuss possible role of chaos in neural networks.

Note that the Hopfield model uses a symmetric weight matrix. In case of a nonsymmetric W there is a possible oscillatory behavior, but at present it has not been successfully applied for pattern recognition purposes.

3.2.3. Brain State in a Box

For this type of networks (see e.g., [43, 45]) the equations of motion have the form

$$\hat{x}_i = f\left(x_i + \beta \sum_{j=1}^{N} W_{ij} x_j\right),$$

where the function f has the form $f(x) = x$ for $x \in [-1, 1]$ and $f(x) = \operatorname{sgn}(x) = \pm 1$ if $x \notin [-1, 1]$, and the matrix W is positive definite. Therefore, the state of the system is always confined to the N -dimensional box $[-1, 1]^N$. This confinement is essential for the model performance. It can be shown, that the corners of the box $|x_i| = 1$ are stable attracting states provided $W_{ii} \geq \sum_{j \neq i} |W_{ij}|$ [45]. This condition guarantees stability of all box corners and also it is sufficient for the matrix W to be positive definite. But for matrices that arise in practice, only a few corners are usually stable. This stability has been the reason for successful applications of the BSB model for data clustering with each stable corner corresponding to a different cluster.

The matrix W can be obtained from the learning data set $x^{(k)}$ by the following iterative scheme

$$W_{k+1} = W_k + \epsilon(x^{(k)} - W_k x^{(k)})x^{(k)T} = W_k(1 - \epsilon P_{x_k}) + \epsilon P_{x_k},$$

where $P_x = xx^T$ is the projection operator onto the x direction. If this procedure were applied infinitely many times to the same vector x, it will converge to the matrix W_∞, for which x would be an eigenvector with the eigenvalue $\lambda = 1$, that is $W_\infty x = x$. If among the training set it is possible to choose several most important dimensions, this procedure will find them, like the principal components analysis. After the matrix W has been obtained, the BSB iterations will exhibit the following results: stable corners correspond to clusters among the training data and enable us to classify subsequent vectors.

3.2.4. Kohonen Networks or Self-organizing Maps (SOM)

This network can be implemented both as multiple-attractor and attractor manipulation networks. The former implementation is slightly simpler, and so we shall consider it here. The main difference between this network and the others in this section is that its learning does not influence its dynamics. The learning is restricted only to preprocessing of initial data.

The network consists of N neurons with competitive dynamics: every active neuron tries to suppress the activities of other neurons, so that eventually only one neuron with the largest initial value $x(0)$ remains active, $x_k(\infty) = 1$, and all other $x_i(\infty) = 0$. An attractor consists of a string of $N - 1$ zeros and a single 1. The number k of this winning neuron marks the recognized pattern or cluster of input vectors. The specific form of the equations of motion is not very important. For example, the equations can have the form $\dot{x}_i = ax_i(1 - \sum_{j=1}^N x_j^2)$. The only purpose of the dynamics is to display, which of the neurons has received the largest input. Often the dynamics is omitted altogether and is replaced by the expression like $k = \arg\max_j x_j(0)$. An example of such a network with detailed description of dynamics can be found, e.g., in [10].

All the recognition and clustering capabilities of these networks lie in the process of preparing $x_i(0)$ from the network input X. The interneuron connections do not depend on the set of patterns to be recognized. But every neuron x_i has n input connections w_{ji}, $j = 1, \ldots, n$, $n = \dim X$, which are used for the preparation of $x_i(0)$. An example is $x_i(0) = R - \sum_j (X_j - w_{ji})^2$. It is convenient to consider the i-th column of the matrix w_{ji} as a vector w_i, then $x_i(0) = R - \|X - w_i\|^2$. It is obvious that the winner is the neuron that has w_i closest to X. If the vectors w_i are normalized, $\|w_i\| = 1$, then $x_i(0) = R - X^2 - \|w_i\|^2 + 2(X \cdot w_i)$. Only the last term depends on i, and hence one can just set $x_i(0) = (X \cdot w_i)$ with the same result of recognition.

This latter form is more "neural:" the neuron receives the signals X_j through the connections with the weights w_{ij}.

If the network has to recognize up to N known patterns $X^{(i)}$, one can set $w_i = X^{(i)}/\|X^{(i)}\|$. The network will check which of the known patterns is the closest match for the network input, and turn the corresponding neuron "on" setting it to 1. If one needs not only the indicator of the pattern, but also the pattern itself, it is possible to generate an n-dimensional output. One introduces the set of output connections w_i^{out} and obtains an output signal $Y = \sum_{i=1}^{N} x_i w_i^{\text{out}}$. Since only one of x_i is nonzero, say, $x_k = 1$, the output of the whole network is equal to w_k^{out}. If it is necessary to reproduce the stored pattern as in the Hopfield model, then it is necessary to set $w_i^{\text{out}} = w_i = X^{(i)}$. But the output may be set arbitrary $w_i^{\text{out}} = Y^{(i)}$. Such a network performs the mapping $X \to Y^{(k)}$ provided $k = \arg\min_i \|X - X^{(i)}\|$. The basin of attraction for the variables x_i corresponds to domains in the space of inputs X. These domains are sometimes called Voronoi or Dirichlet cells: if X belongs to the k-th cell, then the closest of $X^{(i)}$ to it is $X^{(k)}$ [59].

A self-organized map is in fact the way of teaching such an architecture to classify input vectors into a number of self-organizing clusters, or, in other words, the adaptive way of creating the Voronoi cells. Suppose that there are M patterns $X^{(i)}$ which have to be subdivided into at most N clusters. Then one has to choose the input weights w_i in accordance with the data. It is done with the following iterative procedure. The patterns $X^{(i)}$ are successively presented to the network. After determining the winning neuron x_k, its input weights w_k are modified: $\hat{w}_k = w_k + \epsilon(X^{(i)} - w_k)$. That is w_k is slightly shifted towards $X^{(i)}$. After repeating this procedure $\sim 10^4$ times the weights w_k usually form a good clustering for the set of the input patterns [59, 45].

In such a simple version of the algorithm, neighboring neurons may not correspond to neighboring clusters. For this reason Kohonen proposed a simple procedure for clusters positioning among the neurons. All the neurons are arranged in one- or two-dimensional lattice and each of them assigned a neighborhood. Then the training weight modification is made not only for the winning neuron, but also for all its neighbors. This makes neighboring neurons (that is, their w_i) to be closer. Then during the learning the neighborhood size diminishes until it includes only single neuron and (perhaps) its nearest neighbors. Thus the final distribution of neurons corresponds to the distribution of input data clusters. Often, when the input vectors X are two-dimensional, the results of learning are plotted as a distribution of the weight vectors w_i on the plane. It is interesting that if the neurons are arranged in a 1-D lattice, and the distribution of clusters is higher dimensional, then this 1-D lattice tend to fill higher-dimensional domain like fractal Peano curve. Because of this feature, Kohonen networks sometimes are called "topological maps."

Sometimes clustering of vectors is also called "vector quantizing." The applications of such adaptive clustering technique are numerous [59]. The most impressive application seems to be one in which a speech is automatically transformed into a typewritten text (the network has been taught to show which letter corresponds to input sounds) [60, 59].

This idea has been generalized by R. Hecht-Nielsen to self-organized vector mappings [47]. It has been called "counterpropagation network." In it, simultaneous learning of input and output weights is performed from the known pairs $X^{(i)}$, $Y^{(i)}$. The resulting mapping $X \to Y$ is piecewise constant within each Voronoi cell, and for X within a given cell Y is equal to the center of the corresponding cell in Y-space.

3.2.5. Synergetics Computer and Multiple-well Potential Systems

Synergetics computer has been proposed by H. Haken [42]. It is close to the continuous-time version of Hopfield network [50], but with some modifications. For storing patterns it also uses matrix formed according to the Hebbian rule. But instead of using a sigmoid function in equations of motion, higher order terms are added to the potential to stabilize the patterns.

Multiple-well systems also use higher-order terms to construct a dynamical system with stable fixed points at necessary places. This can be done e.g., by a proper choice of the potential in a form $U(x) = \sum A_i V(x - x_i)$. The problem of mutual interference of states can be solved by a proper choice of all parameters. This way of constructing the recognizing dynamical system was proposed in the beginning of 90's. But it seems to have nothing in common with biology and real neurons, and slightly resembles the method of radial basis function.

We have described here only the most common types of multiple-attractor networks. For their "classical" versions, complex dynamics is not relevant, though it has been used in attempts to overcome some shortcomings like false memories of Hopfield networks.

3.3. Attractor Manipulation Networks

3.3.1. A Simple Example

Let us consider an equation

$$\dot{x} = -x + X.$$

This is a gradient system with the potential $U(x) = x^2 - Xx$. That means that the input changes the system potential. For any initial value $x(0)$ the final state is $x(\infty) = X$. Therefore, this system performs the mapping $Y = X$.

Certainly, other functional dependencies can be organized this way. For example, the system $\dot{x} = -x + \tanh(X)$ provides the sigmoid function output. Similar (and probably more interesting) results can be achieved by a potential with an infinite well,

$$\dot{x} = -\frac{ax}{1 - x^2} + X.$$

The resulting mapping is

$$Y = \frac{2X}{a + \sqrt{a^2 + 4X^2}},$$

that is, almost sigmoid function. Note, that for small a this function is close to the step function $Y = \text{sign}(X)$, and therefore can be used for classification or recognition purposes, similar to multiple-attractor networks.

However, to be able to interprete $x(\infty)$ as a value of the function $F(X)$, the dynamical system for every X must have a single attractor, which is a fixed point. If there are many attractors, F may have several values for a given X, and if the attractor is not a fixed point, there will be no limiting value $x(\infty)$. But usually neural network designers try to avoid such situations, and use special theorems which guarantee the existence of a stable fixed point, e.g., the famous Cohen-Grossberg theorem [39]. Usually the existence of a Lyapunov function for network equations and absence of complex temporal behavior is considered as a great virtue.

Some of the networks can be implemented both as multiple-attractor and attractor manipulation networks, an example is the Kohonen network described in the previous section. In the latter case the input X has to act all the time, and be a set of parameters in the equations of motion. Another very famous example of attractor manipulation is S.Grossberg's adaptive resonance theory network and W. Freeman's model of olfactory system. But the latter shows complex dynamics that will be discussed in Section 4.

3.3.2. Adaptive Resonance Theory Networks (ART)

Like the Kohonen network, a basic ART network has a vector X as its input and a cluster indicator as its output. See [18] for a description of generalizations of the ART network. According to [18], proper functioning of such networks requires rather thorough tuning. As a result, what we describe here is simply a brief review of some basic ideas, and not a recipe for building an ART network.

The network has 3 layers: the input layer (where the signal X is fixed), the modified input layer F_1 (with elements denoted by x_i), and the output layer F_2 (with the elements y_j). As in the case of SOM, this network should represent a cluster, only one y_j should eventually be nonzero. The information flow is as follows: $X \to F_1 \leftrightarrow F_2 \to$output.

The equations of motion in the most convenient and compact form are presented in [80]. We shall cite the ART1 version, developed for clustering of binary input patterns, when components of X can take values only 0 or 1. The dynamics for F_1 is governed by the equation

$$\begin{aligned}\frac{dx_i}{dt} &= -x_i + (1 - A_{11}x_i)\Big(X_i + \sum_j T_{ij} f(y_j)\Big) \\ &\quad -(A_{12} + A_{13}x_i)\sum_j f(y_j).\end{aligned}$$

The first term on the right-hand side guarantees that $x \to 0$ when no input is presented. The second term ensures that in the presence of an input, the growth of x is limited. The sum $X_i + \sum_j T_{ij} f(y_j)$ means, that x may gain large enough amplitude only if the input X matches the stored pattern in the top-down ($F_2 \to F_1$) connections T—the "expected input", corresponding to the present activity of F_2. The third term enables to suppress the present x pattern when F_2 layer is active if there is no matching between input and expectation. The function f is a sigmoid-type function: $f(x) = 0$ for $x \leq 0$ and $f(x)$ grows and saturates at 1 for $x > 0$.

The equations for F_2 layer look similar,

$$\begin{aligned}\frac{dy_j}{dt} &= -y_j + (1 - A_{21}y_j)\Big(f(y_j) + \sum_i B_{ij} f(x_i)\Big) \\ &\quad -(A_{22} + A_{23}y_j)\Big(\sum_{k \neq j} f(y_k) + r_j\Big).\end{aligned}$$

The second term, like in SOM, excites the ncdes according to bottom-up connection matrix B_{ij} along with self-excitation, while the third term describes suppression of the neuron activity, and special reset term r, which will be discussed later.

The dynamics of learning for the matrices B and T is rather simple and resembles that of SOM:

$$\tau \frac{dT_{ij}}{dt} = f(y_j)(f(x_i) - T_{ij}), \qquad \tau \frac{dB_{ij}}{dt} = f(y_j)\Big(\frac{f(x_i)}{1 + |F_1|} - B_{ij}\Big),$$

here $|F_1|$ is the activity of the F_1 layer (the number of active neurons). These equations ensure that learning will occur only for connections, for which the neuron y_j is active, and the stored pattern will be that generated by F_1. This is the well-known Hebbian learning.

Now let us consider the operation of this network. An input pattern X activates the F_1 layer, and it, through the matrix B, activates some pattern in the layer F_2 with one of y_j in the on-state. The pattern stored in T which corresponds to $y_j = 1$ generates another input (top-down) to F_1. If these two inputs match, the network remains in this state and adjusts the matrices B and T. But if these patterns are not close enough, then the search mechanism is started. First, the reset variables r are activated,

$$\tau_r \frac{dr_j}{dt} = f(y_j)(Df(v|X| - |F_1|) - r_j).$$

Here v is the 'vigilance parameter', which quantifies the closeness of input and expected patterns, and $|X|$ and $|F_1|$ denotes the number of "on" elements in the layers X and F_1 respectively. After r_j has been activated, it suppresses the active F_2 neuron y_j (for the best network performance it should stay suppressed until search is ended). After that the top-down influence interrupts, pattern in F_1 restores and the network begins to activate another neuron in F_2. If its pattern stored in T also does not match, this node is also suppressed and the search continues until appropriate F_2 neuron is found. It may be a "free" neuron, for which no expected pattern exists, in this case a new cluster will start to form. Or the layer F_2 may be exhausted, then the network fails to classify the input.

S. Grossberg used the phrase adaptive resonance for this matching between F_1 and F_2 patterns. Here only an outline of this approach is mentioned. There are several implementations of this idea: ART1 for binary patterns, ART2 for analog patterns, and ART3 for the case in which the layers F_1 and F_2 are symmetric. In every implementation there are different versions. Note that proper network operation requires accurate tuning of all parameters, for detailed descriptions see [19, 20, 21, 39, 18, 80].

The interesting feature of ART1 and ART2 architectures is their ability to be activated from both sides, that is the layer F_2 can in principle be turned into the input layer. Then two ART2 modules can be combined such that the first module receives input and generates its category, while the second module receives category to its F_2 and generates analog output signal in its F_1. This approximation network has been called ARTMAP [22].

3.3.3. Hopfield-Tank Optimization Network

This network was proposed in [51]. The equations of motion for the network coincide with that for continuous-time Hopfield network described in [50],

$$\frac{dx_i}{dt} = -\frac{1}{\tau}x_i + \sum_{j=1}^{N} w_{ij} f(x_j) + I_i, \tag{6}$$

where f is a sigmoid-type function, w is the weight matrix and I_i is the firing threshold of neuron i. Equations (6) describe a steepest descent dynamics $\dot{x} = -\nabla E$ for some energy functional $E(f, w, I)$, and the final state should be a fixed point corresponding to minimum of E. Learning for this network means constructing the matrix w and the bias I in a special way, to place the minimum in a desired point of the phase space. The weaknesses of this method include the facts that it is hard to guarantee the absence of spurious attractors. Later in Section 4 we shall consider, how chaos helps to avoid spurious states.

For the traveling salesman problem (TSP), Hopfield and Tank proposed a special form of the energy function depending on the distances d_{XY} between the towns to be visited. Therefore, we come to the general form of parametric attractor manipulation with d_{XY} being the parameters. Due to the presence of false attractors, the solutions found by the network could not be optimal but, according to [51], close enough to it.

3.3.4. Other Networks

There are other attractor manipulation networks. For example, the well-known Boltzmann machine falls into this category, since during learning some of its neurons are "clumped" and play the role of parameters rather than dynamic variables. Freeman's model of olfactory system [35] also falls into this category. Some networks of this type will be considered in the next section.

4. Complex Dynamics as an Attempt to Improve ANN

Artificial neural networks appear to be universal tools for approximations, pattern recognition, etc. Nonetheless, difficulties may arise during their application. For functional networks, for example, problems arise when one tries to approximate very complex functional dependence: a too simple network can not give proper approximation, and a too complex one amplifies noise. This problem is called "overfitting" and it is usually solved by introducing a special "second level" structure of a network called the modular and ensemble networks [82].

For conventional dynamical neural networks considered above, the problems that arise most often are (1) convergence to wrong attractors (false memories), (2) too slow convergence to attractors and (3) failure to reproduce the activities of the real brain.

Dynamics indeed has a potential for information processing. For example, an interesting idea about its application has been proposed by A.Dmitriev et al. [4]: an image is stored as a long cycle of a dynamical system, and each step of the cycle corresponds to an image pixel.

Another type of networks with complex dynamics emerged from the analogy between neuron or small groups of neurons and an oscillator. Such a neural network is a lattice of coupled oscillators. It has been shown that information may be stored and processed in the phases of the oscillators. Such an oscillatory network may implement the principles of a Hopfield network [52] or other types of networks, e.g., [69, 48].

The idea that neural networks can work in chaotic regimes has also been proposed in a number of papers. A good review of the basic approaches in this field for early 1990s can be found in [34, 93], and we shall not repeat all that has already been reported. We shall describe several models focusing on the basic directions of studies in this field. In order to ascertain the usefulness of chaos, it is instructive to pay attention to the following properties of chaos [28, 40]:

1. *Local instability.* Local instability might be useful for preventing false memories and staying off trajectories that lead to undesirable locations in the phase space. This effect of chaos may be useful only during a transitional period while the trajectory converges to a "true memory." After such a transition, the dynamics should again become stable.

2. *Creation of information.* This may be the most attractive property of dynamical chaos. The fact that a chaotic system behaves unexpectedly can be used to search a new way for solving a problem. In other words, chaos can help in exploration of possibilities. However, modern neural network architectures can not use this property, because their desired behaviors are predictable: always the same output for a given input. There were attempts to use chaotic signals in algorithms for random minimization during learning of multilayer perceptrons, but deterministic methods had the upper hand. Neural networks capable of exploration during their learning are still to be found.

3. *Wandering along attractor.* This property is closely related to instability. There were attempts to use it as a tool for memory search with the help of chaotic attractor. The trajectory wanders between images, and the idea was to use this property for matching input data with the stored images. Several experiments have been done and preliminary results obtained, but the corresponding pattern recognition algorithm has not been developed.

4. *Resemblance with complex behavior of the brain.* A number of attempts have been made, mainly by Freeman and his colleagues [34, 31, 32, 99] to create a neural network with the structure resembling that of a part of the brain, mostly the olfactory bulb. The behavior of the brain may strongly change in time, and chaotic systems also can demonstrate a variety of behaviors. The behavior of models used resembled experimental signals from the brain.

5. Finally, chaos may be used without any special role: we can create a multiple-attractor system with periodic or chaotic attractors instead of a fixed point [10]. The kind or number of a chaotic attractor can be considered as a result of recognition, though a special decoding subsystem must be added to transform a chaotic signal into the deterministic network answer.

In the following section, we shall consider several neural networks with chaos.

4.1. Transient Chaos vs False Memories

4.1.1. Chaotic Modifications of the Hopfield-Tank Model

One of the drawbacks of the Hopfield-Tank model is that the dynamics can be trapped in local minima. To overcome this difficulty, a number of workers [87, 62] have used transient chaos and noise. Recently Kwok and Smith [62] have presented a unified approach to such neural networks. It is worth reporting one of the examples to illustrate the basic ideas involved.

Chen and Aihara [24] have used transient chaos in a network described by

$$\begin{aligned} x_i(t+1) &= kx_i(t) + \alpha\left(\sum_{j=1,j\neq i}^{N} w_{ij} f(x_j(t)) + I_i\right) - z(t)(f(x_i(t)) - I_0) \\ z(t+1) &= (1-\beta)z(t) \end{aligned} \tag{7}$$

where $z(t) \geq 0$ is the self-feedback term and $0 \leq \beta \leq 1$ is the damping term. These equations of motion can be obtained from the equations of motion similar to (6) by time discretization. Due to the last term with $z(t)$ the system can be chaotic. The dynamics starts with a large value of $z(t)$ to ensure the existence of chaos, and then $z(t)$ is reduced according to (7) and the system can converge to the attractor. So, the main purpose of using chaos in this type of work is to overcome the difficulties associated with spurious states.

Another network with transient chaos for optimization problems has been proposed in [23]. The network is again a mapping, but now with a delay

$$x_i(t+1) = \sum_{j=1}^{N} w_{ij} f(x_j(t)) + I_i + g(x_i(t) - x_i(t-1)), \tag{8}$$

where f is again a sigmoid function and $g(x) = axe^{-b|x|}$. In a stationary state the last term vanishes and therefore the fixed points of (8) coincide with that of usual Hopfield-Tank model (6).

The term g makes the equations of motion more complex and ensures the chaotic wandering. Due to it the spurious stable states of the original Hopfield-Tank network in numerical simulations became unstable, though trajectory sometimes spent some time near them. To ensure convergence to the global minima, authors of [23] proposed special control scheme for parameters a and b. The experiments showed that the network successfully solved the TSP problem.

Another modification of the Hopfield-Tank model with additive chaos or noise was studied in [44].

4.1.2. The Hopfield-type Network with Transient Chaos

This network was proposed in [55]. The idea is to replace a simple threshold neuron with an one-dimensional dynamical system $x_{t+1} = f(x_t, \mu)$—a neuron with its own internal dynamics.

The resulting network of mappings is controlled by the system's energy E via the parameter μ. If the energy is high, the dynamics of the mappings is chaotic, and when the energy becomes low, the trajectory of the mapping tends to one of two fixed points, close to ± 1. Therefore, during transient phase the system's dynamics is chaotic, and close to the energy minimum it turns stable. The equations of motion for each "neuron" has the form

$$x_i(t+1) = f(x_i(t), E_i), \tag{9}$$

where

$$F(x, E) = \{K(E)(x + |E|)\} \bmod 2 - 1, \qquad K(E) = 2(1 + |E|)^{-1}. \tag{10}$$

(chaotic behavior for $|E| < 1$ and regular otherwise). The parameter E is called the local energy

$$E_i(t) = \lambda \sum_{j=1}^{N} w_{ij} x_j(t).$$

The matrix w, as in the original Hopfield model, is constructed according to the Hebbian rule (5). The definition of the local energy E_i involves the parameter λ, which describes neuron interactions: when $\lambda = 0$—the network splits into N independent mappings, and when $\lambda = \infty$, one obtains the usual Hopfield model.

Numerical experiments show, that for large values of λ the system behaves as the Hopfield model, but for moderate values (for the specific example considered in [55], $\lambda \leq 5$) there is a difference: almost all the false memories become unstable, and the system either remains chaotic for a very long time or converges to one of the stored images. So, in this case chaos helps in getting rid of false memories.

4.1.3. Coupled Map Lattice with Nonstationary Synchronous Clusters

These networks were described in [54]. They are based upon the globally coupled map lattice with the equations of motion

$$x_i(t+1) = (1-\epsilon_i)f_i(x_i(t)) + \frac{\epsilon_i}{N}\sum_{j=1}^{N} f_j(x_j(t)), \qquad (11)$$

$$f_i(x) = \alpha_i x^3 - \alpha_i x + x, \qquad 2 \le \alpha_i \le 4. \qquad (12)$$

There are two versions of this system: α-version, when $\epsilon_i = \epsilon$, α_i are different, and ϵ-version, when all $\alpha_i = \alpha$, ϵ_i are different. Bifurcation diagram for the mapping (12) resembles that for the logistic map. The encoding and decoding are very simple, all positive values $x > 0$ are associated with the value $+1$, and negative values with -1.

The idea of the CML-based pattern recognition arose due to the fact that in the CML (11) with $\alpha_i = \alpha$ and $\epsilon_i = \epsilon$ there is a domain of parameters (α, ϵ), where the network splits into synchronized clusters corresponding to periodic behavior. For these parameters the system possesses rather high "information preservation," that is initial data strongly influence the subsequent behaviors, and there is high correlation between $x(t)$ and $x(0)$. For larger α or smaller ϵ in chaotic state this property is lost.

The idea is to define a local energy functional $E_i = -x_i \sum w_{ij} x_j$ and use chaos instead of annealing to break unwanted correlations. The matrix w is formed according to the Hebbian rule using the patterns $\xi^{(k)}$ to be stored, $w_{ij} = \sum_k \xi_i^{(k)} \xi_j^{(k)}$. Then in the α -version, the dynamics of x is augmented by the dynamics of α:

$$\alpha_i(t+1) = \begin{cases} \alpha_i(t) + (\alpha_i(t) - \alpha_{\min})\tanh(\beta E_i) & \text{every 16 steps,} \\ \alpha_i(t) & \text{otherwise.} \end{cases}$$

The value of $\alpha_{\min}$ corresponds to the clustering phase. This causes diminishing of α, that is entering more ordered phase when the local energy is low enough.

Computer experiments show that this system works as an associative memory, and its memory capacity is about $0.18N$. Similar characteristics are obtained for the ϵ-version of the algorithm.

Another version of pattern recognition with the transition chaos→order has been proposed in [67].

We note that this network does not use chaos for recognition. It uses chaos only for some short transient period although the normal dynamics of the system is non-stationary.

4.1.4. What if Transient Never Ends or a Novelty Filter

There is always a possibility that chaotic transient states, under some combinations of input parameters, can give rise to a chaotic attractor or a very long transient period. Skarda and Freeman [85] supposed that such a state can mean "I don't know," that is, a neural network faces something that has not been learned (see also [55]). Such a state, like the inconsistency in bottom-up and top-down patterns of ART maps, can be used in principle as a novelty filter to initiate the learning phase. To our knowledge examples of networks with such filtering have not been published.

4.2. Chaos in Memory Search

When a trajectory moves along a chaotic attractor, it moves sequentially from one part to another. If we associate various parts of the attractor with different patterns, then the trajectory will wander between them. In principle, this wandering can be used for the recognition or association purposes: if a trajectory spends most of its time near one of the patterns, then the latter can be considered as "recognized", and if in the sequence of visited patterns there are stable combinations, those patterns may be considered as "associated" with one another. Note that sequences of patterns can be stored into a Hopfield-type networks. There is a possibility that chaos may help vary these combinations to learn new ones or to allow one pattern to participate in a number of associations simultaneously. These ideas have not been implemented completely, but some preliminary results have been obtained.

To study the linking of stored memories with one another, Tsuda [93] proposed a model that basically resembles a Hopfield-type network. Initially, patterns are stored by the Hebbian rule, but afterwards the connection matrix is dynamically modified. It has been shown that association of patterns with one another takes place, but there is not enough control over the process.

Another example of a model with associative dynamics in chaos has been proposed in [1] and references therein. The equations of motion had the form

$$\begin{aligned} x_i(t+1) &= k_f x_i(t) + \sum_{j=1}^{N} w_{ij} f(x_j(t) + y_j(t)) \\ y_i(t+1) &= k_r y_i(t) - \alpha f(x_i(t) + y_i(t)) + a_i, \end{aligned}$$

where $0 \leq x, y \leq 1$, $f(x) = 1/(1 + e^{-x/\epsilon})$, and patterns are stored in the network with the help of the Hebbian rule, $w_{ij} = \sum_k (2\xi_i^{(k)} - 1)(2\xi_j^{(k)} - 1)$.

The results of numerical experiments show that trajectories of the system visit neighborhoods of the stored patterns, and after a coarse-graining transformation some of the states reproduce the stored patterns. But the network does not give any definite pattern that can be considered as the resulting output. The presented results show some dependence on the behavior of initial conditions, but it is not clear, whether they correspond to (i) different attractors, (ii) transient processes or (iii) insufficiently long trajectories to establish a good statistics.

The chaotic wandering between stored patterns has also been studied in [26] and called "chaotic memory scanning."

A different idea for using chaotic regimes for recognition purposes has been proposed in [86, 87, 88, 89]. It has been observed, that patterns can be recognized by the shortest time required for full or phase synchronizations of an unknown pattern with known ones. The strength of this approach is that the procedure is general and can be applied to chaotic as well as non-chaotic neural networks. In this approach, one does not need fixed points or stationary final states for pattern recognition. The weak point is that the dynamics of known patterns needs to be run in parallel with that of the unknown pattern. It has also been found that virtual basins of attraction can be created around fixed points by root finders. What is good about this approach is that it utilizes the dense set of periodic orbits of chaotic NNs and thereby increases the capacity enormously. The work that needs to be done is to find ways to map patterns to state variables of the network.

4.3. Chaos Instead of a Fixed Point

As we have mentioned in the beginning of Section 4, the multiple-attractor and attractor manipulation networks can be implemented with attractors other than fixed points. One of the simplest ways to construct a multiple-attractor network is described in [10]. For attractor manipulation networks there is no such simple example. This may be because a complex bifurcation structure of chaotic attractors may cause severe difficulty in distinguishing attractors for different parameter values. Nonetheless, one of the most famous chaotic networks belongs to the attractor manipulation class. This is the Freeman's model of olfactory bulb.

The studies of olfactory system have been the primary goal of W. Freeman and his colleagues for several decades, see [99, 31, 32, 30, 85] and references therein. After some years of biological studies of olfactory bulb, they concluded that studying only the structure of neurons and their connections is not enough to understand the neural mechanisms responsible for olfaction. For this reason they developed a number of mathematical models for information processing in olfactory bulb. The dynamics of the models are in qualitative agreement with the experimental EEG measurements, and is chaotic.

The model is rather complex. Each "memory unit" is described by about 10 second-order differential equations, which describe the "specialization" of neurons within every unit. All equations have similar form $\ddot{x}_i + A\dot{x}_i + Bx_i = G_i$, where the right hand side terms are different for different types of neurons [99]. Some of the G_i include input terms X, while others have delayed input (dependent on past values of x). There are neurons responsible for connections with other memory modules, and for them G_i include the term $\sum K[i,j]Q(x[j])$, where Q is a sigmoid-like function and $x[j]$ is similar "connection" neuron from the j-th memory module.

Information in this network is stored into the connections $K[i,j]$. They can take only two values, $K_{\min}$ and $K_{\max}$. Initially, all connections are set to $K_{\min}$, and to store pattern, for which modules i and j are both "active", the corresponding $K[i,j] = K[j,i]$ is set to $K_{\max}$ (Hebbian rule).

The network works as follows: When there are no external signals, the network oscillates chaotically on an attractor. If an external stimulus is presented to the network, the system stabilizes onto individual parts of the attractor. To explain the main idea, one can consider a multilobed attractor such as the Lorenz attractor which has two lobes as the network's "basal activity state." By applying an input, the network can be stabilized onto one of the two lobes. The dynamics would still be chaotic but confined to one of the lobes only. Then the presence of the trajectory at this lobe can be decoded into the network output, e.g., with the help of local time averages.

Other chaotic models of olfactory bulb also exist, e.g., [6], though they are not in very good agreement with biological experiments.

4.4. Recurrent Neural Networks as Generators of Chaos

There are works that consider neural networks just as models for some biological phenomena or convenient forms of dynamical systems. There is no question of computation, approximation or associative memory, e.g., [34, 2, 3, 61, 70] and some other. This class of neural networks falls out of our scope. Indeed, it is not hard to construct a neural network, e.g., in a form of recurrent perceptron (Section 3), with chaotic dynamics. Such works do not answer the question about the role of chaos in information processing. Nonetheless, if there is such a role, those neural networks may serve as a convenient generators of chaos [75].

4.5. What's Wrong with Chaotic Networks?

We have mentioned only some of the works related to chaotic neural networks just to illustrate major directions of studies. A common feature of all chaotic networks is that they are not used in practical applications. The only exception is the experiment described in [100]. In contrast, multilayer perceptrons are widely used, while chaotic networks have remained only as objects of theoretical studies for about 15 years. What is the reason for this? From our point of view, the reason is the way in which neural networks are currently used. Current uses of neural networks may be called "isolated computations." A neural network's task is only to generate definite, always the same output for a given input. Chaotic dynamics, which is unstable, can only make such a computation unreliable. Therefore, there is no apparent room for chaos in such a scheme.

In contrast with such neural networks, the brain always works as a part of the body. It is involved in continuous processing of information coming from the outside world, and it guides the body to perform actions that change the environment. Therefore, the brain operates as a part of a closed loop: brain—actions—world—sensing—brain. It is possible that accounting for the embodiment of the brain can explain the advantages of operating in chaotic mode. Also chaos may just emerge as a consequence of positive feedback in the aforementioned loop.

Note that the problem of "embodiment of intelligence" has been intensively discussed in Artificial Intelligence during the last 15 years, see [73] for a review. The related approach called "behavior based robotics" or "embodied cognitive science" led to a number of efficient practical solutions and new theoretical concepts. Moreover, with examples of small robots controlled by a rather simple neural networks, it has been shown that closing the loop through the world may lead to a very complex, probably chaotic behaviors [73].

Another natural source of complex temporal behaviors may be special implementations of neural networks: using ensembles of coupled oscillators to perform computation, e.g., [52, 48, 69]. In Section 7 we shall discuss this point in more details.

5. Closing the Loop: Chaos in a Combination of Controlling Neural Network and Controlled System

As we said in Sections 3, 4, one of the simplest ways to obtain chaos is to take a functional network, say a feed-forward multilayer perceptron approximating equations of motion of a chaotic system, and feed its output back to the input. The result is a chaotic dynamical system. However, it does not perform useful information processing. A useful task can arise if we place between output and input of a network a system that needs controlling (often called the "plant"). In this section we present some rather simple examples that demonstrate how chaos can appear in a controller coupled with a controlled system. Chaotic signals can be registered at any part of the combined controller-plant system. This notion may partially explain the appearance of chaos in the activities of the real brain.

Here we are referring only to chaos emerging in the course of information processing. In the next section, we will consider the importance of chaos in the learning process of a controller-plant system.

Let us consider a dynamical system

$$\dot{x} = \lambda x + f, \qquad \lambda > 0. \tag{13}$$

For $f = 0$ it has an unstable equilibrium point $x = 0$. Suppose that we can control this system by applying the "force" f at the discrete moments of time $t_k = \tau k$. Then the force remains the same until the next switching. The absolute value $|f| = f_0$ always remains the same, we can only change its direction. Our goal is to keep the trajectory in the vicinity of the point $x = 0$. So, at every t_k we know $x(t_k)$, and we have to make a decision regarding the direction at which the force should be applied.

This is a simple problem. Let us write $x_k = x(t_k)$. Since $f_k = f(x_k)$ remains constant until t_{k+1}, (13) gives: $x_{k+1} = e^{\lambda\tau} x_k + (e^{\lambda\tau} - 1)\frac{f_k}{\lambda}$. It is easy

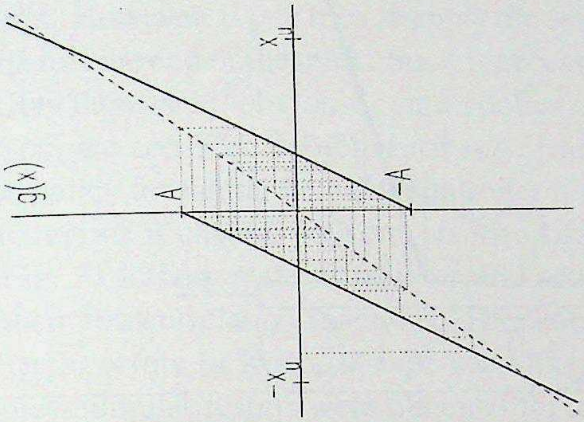

Figure 1. Mapping resulting from discrete control of unstable fixed point and an example trajectory.

to check that the choice $f_k = -f_0 \operatorname{sgn}(x)$ solves the problem, and we obtain the following one-dimensional mapping

$$x_{k+1} = g(x_k), \quad g(x) = \begin{cases} e^{\lambda\tau}x - A, & x \geq 0 \\ e^{\lambda\tau}x + A, & x < 0 \end{cases}, \quad A = \frac{\left(e^{\lambda\tau} - 1\right) f_0}{\lambda}. \quad (14)$$

The plot of $g(x)$ is shown in Fig. 1. It can be seen that the trajectory remains near the unstable point provided $|x(0)| < f_0/\lambda$. Since $dg(x)/dx = e^{\lambda\tau} > 1$, a chaotic attractor is born with the Lyapunov exponent equal to λ.

The control of the system can be performed by a "network" with a single threshold neuron that receives input x_k and generates the signal ± 1 , showing the direction of the force. As the attractor is chaotic, the sequence of the neuron outputs will look random. The source of this randomness is a discrete control of an unstable equilibrium.

This example explains the main idea, but it has two obvious shortcomings: (i) there is no learning, and (ii) there is no true need for the use of a neural network. Let us consider more complex examples related with discrete controls. Numerous examples of such problems can be found, e.g., in the literature on machine learning [65].

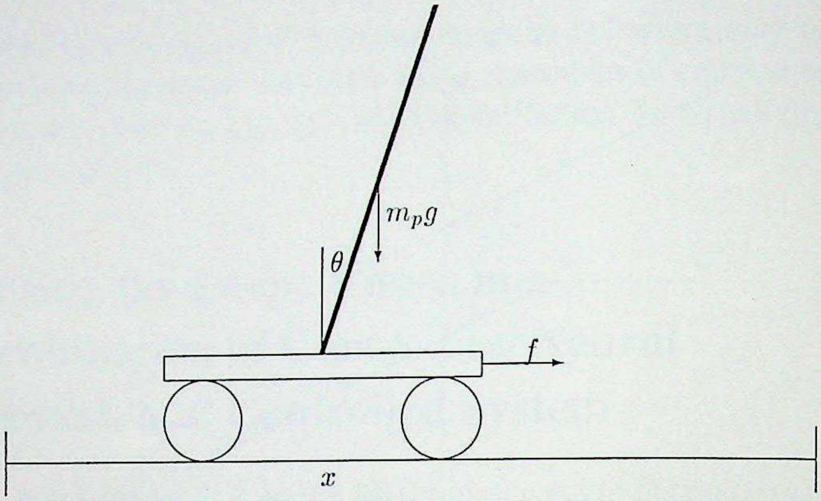

Figure 2. The cart-pole balancing task. Controller should choose the proper direction for f after each time interval τ such that the angle θ for the pole will remain within $[-\theta_{\max}, \theta_{\max}]$, and the cart never hits the ends of the track, $-x_{\max} < x < x_{\max}$. In the beginning the cart is positioned at the middle of the track $x = 0$ and the pole is set at some angle θ_0 which is within the admissible limits.

The easiest generalization is an inverted pendulum that need to be kept close to its highest point. However, it can be shown that this problem reduces to the above example — the unstable manifold of the fixed point of the inverted pendulum is one-dimensional.

A more interesting problem is that of cart-pole balancing, one of the well-known benchmark problems in machine learning [64, 12, 46]. There is a cart that can move along the line from $-x_{\max}$ to $x_{\max}$. A pole is attached to the cart with one end such that it can rotate in the vertical plane parallel to the line of motion of the cart. If the pole is set almost vertical, while falling, it moves the cart. If one pushes the cart, the push affects the pole dynamics as well. That is, by moving the cart, one can change the position of the pole. The state of the cart-pole system is determined by x (coordinate of the cart), $\dot{x}$ (velocity of the cart), θ (inclination angle of the pole from the vertical), and $\dot{\theta}$ (angular speed of the pole), see Fig. 2. The task of control is as follows: After every time interval τ, the controller receives the values of the cart-pole state variables x, $\dot{x}$, θ , $\dot{\theta}$. The controller can apply a force equal to $\pm f$ to the cart that acts during the next τ-interval. The task is to keep the angle θ within the limits $[-\theta_{\max}, \theta_{\max}]$, and the position of the cart x within $[-x_{\max}, x_{\max}]$.

At the first glance this problem seems equivalent to the control of the inverted pendulum. However, if one tries to apply the same control algorithm, the cart very soon hits the end of the track. So, it is necessary to control the cart position as well, but there is no "obvious" control policy that should be learned. In the next section, we consider the details of such learning. Here, we only mention that the policy for neural network controller has been obtained [77], and the resulting regime is chaotic with one positive Lyapunov exponent. We registered (see Fig. 3) activity patterns from several neurons of the controller and calculated their autocorrelation functions. They look chaotic, though the neural network architecture (a Kohonen-type network) does not possess any complex dynamics. Similar results were obtained for another model control task — stabilization of an unstable chemical equilibrium [77].

So, chaos may arise in complex feedback loops where a neural network plays the role of a learning controller. One may ask: can chaos be useful for the information processing or for learning? The results of our models show that the answer is yes, but to explain it we need to describe the idea of reinforcement learning.

6. Chaos and Reinforcement Learning

6.1. What is Reinforcement Learning?

In most books on neural networks two types of learning are considered, supervised and unsupervised learning. If for every training input X the correct output Y is known, and this knowledge can be used for updating the weights of the network, the learning is called supervised learning or learning with a teacher. Examples of supervised learning include all functional and most dynamical networks. If learning proceeds without a teacher that can provide the correcting signals, then it is called unsupervised learning. For examples of unsupervised learning see, for example, Kohonen and ART networks. In addition to these two types of learning, an intermediate situation is possible when some evaluation of the network performance can be done, but the correct answer is unknown. Usually, such an evaluation of performance comes in the form of a scalar 'reward' r : for success $r > 0$, for failure $r < 0$, and for neutral case $r = 0$. The corresponding learning is called reinforcement learning or learning with a critic. This type of learning is rarely used in traditional neural networks. However, it is very valuable in situations where one knows the

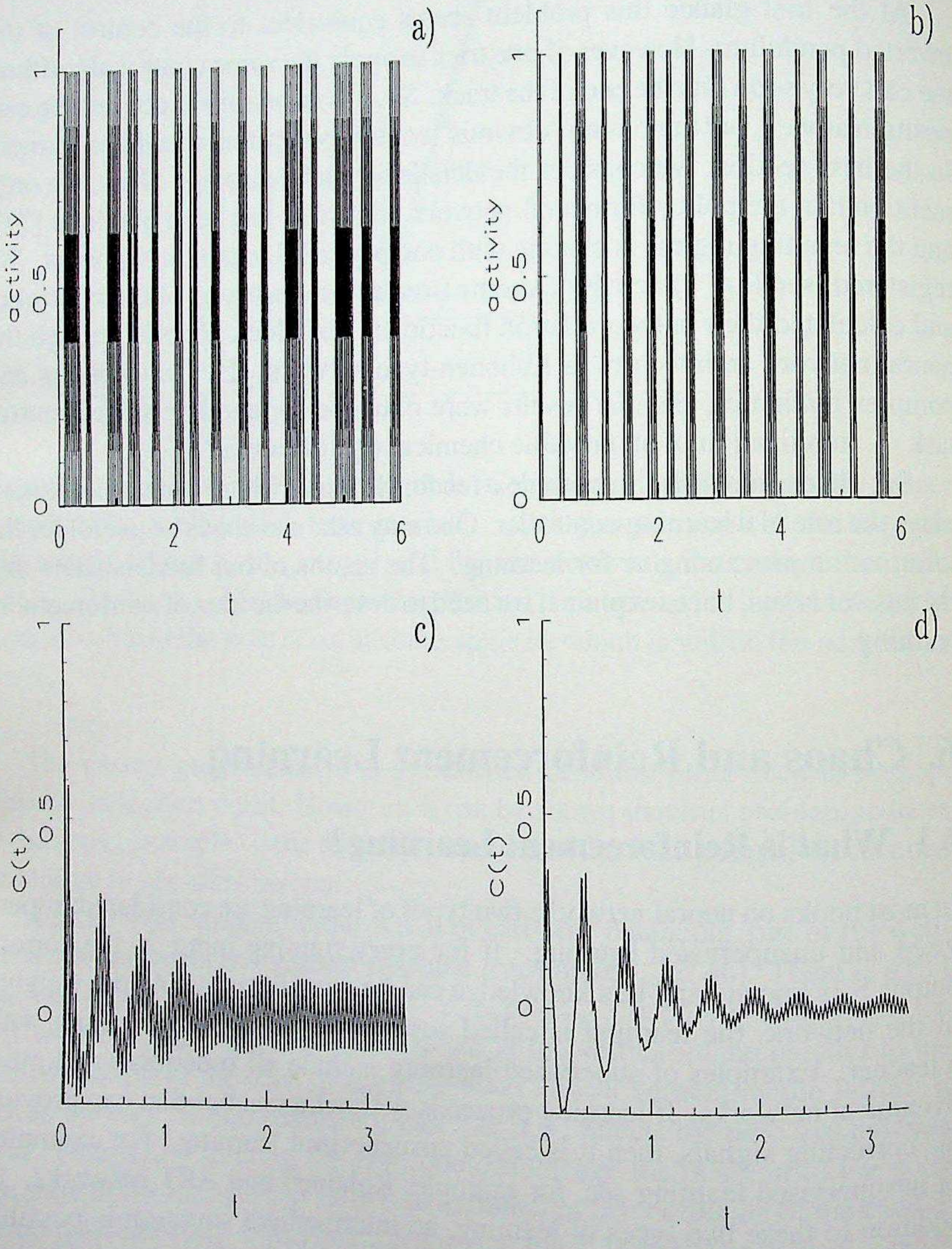

Figure 3. "Encefalogram" for the cart-pole control. Panels a and b show the temporal activity for two of the neurons of the controlling network (1 when control was performed by this neuron and 0 otherwise). Panels c and d show corresponding autocorrelation functions. Plots show both determinism and randomness.

final goal but does not know the correct way to achieve it. An example is the cart-pole balancing problem in which nobody can tell what direction of force the controller must choose at an instant although it is easy to detect a failure when the pole falls or the cart reaches the end of the track. It may be pointed out that the reward may be delayed. An example is the game of chess: the reward comes only after many moves, and it is often hard to tell which moves were right and which ones were wrong. The problem of determining the role of every action in attaining the final outcome is called the "credit assignment problem."

It is quite likely that highly organized living organisms make use of some kind of reinforcement learning. These organisms have a rather sophisticated source of reward signals such as the feelings of pain and delight. The existence of such a reward system is advantageous to these organisms. For mobile robots, similar sources of reward are also used [73], they are called "value systems."

In view of the present trends in artificial intelligence and autonomous robots, methods of reinforcement learning are expected to become increasingly important. The trouble is that there is no general theory for such a learning: the concept is not very specific and can be applied to different situations and systems. However, much progress has been made in a class of problems known as Markov Decision Processes (MDP) or Dynamical Programming (DP).

The formal description of MDP includes (i) an environment which is characterized by its state s, (for example, a system to be controlled, e.g., the cart-pole, where $s = \{x, \dot{x}, \theta, \dot{\theta}\}$), and (ii) an agent, (controller) who can perform action a (in the cart-pole example, the action is the application of the force $\pm f$). For the sake of simplicity, we shall assume that both s and a are discrete. The agent receives information about the state s_t at time t, and undertakes an action a_t, which influences the environment. After each action the environment changes its state, the agent receives information about the next state s_{t+1} and a scalar reward signal r_{t+1}. The rule, according to which the agent chooses its action $\pi(s, a)$ is called policy. Usually, $\pi(s, a)$ is the probability of taking the action a. Deterministic choice means that for only one of the actions the probability is nonzero.

The task of the agent is to work out an optimal policy which brings maximal total reward during a large or infinite number of actions. In the latter case the sum $\sum_{k=1}^{\infty} r_k$ may be infinite, and hence the discounted total reward $\sum_{k=1}^{\infty} \gamma^{k-1} r_k$, $0 \leq \gamma < 1$ (usually $|r_k|$ is bounded) is used.

Up to this point, the situation looks very general. The restricting assumption for theoretical study is that the dynamics of the environment is Markovian. Let $P = P(a, s, s')$ be the probability of change of state of the environment from s to s' after the action a has been made, and the value of reward in this case be $R = R(a, s, s')$ (r_k is always equal to R with the proper arguments). It is assumed that the transition probability P and reward R do not depend on previous states of the environment, on the previous actions of the agent or on time. Therefore, the dynamics of the environment can be described in terms of controlled Markov chain. Later, we shall also assume that this chain is ergodic.

If all P and R are known, then it is possible to estimate the expected discounted reward for the state s, action a and the selected policy π by averaging rewards for all possible paths along the Markov chain,

$$Q^\pi(s,a) = \sum_{s'} P(a,s,s')\Bigg(R(a,s,s') + \gamma \sum_{a'} \pi(s',a') \sum_{s''} P(a',s',s'') \\ \times (R(a',s',s'') + \dots)\Bigg).$$

The $Q^\pi(s,a)$ are called action values. Since we have assumed that π, P and R do not depend on time, the same Q^π appear on the right hand side, so that

$$Q^\pi(s,a) = \sum_{s'} P(a,s,s')\Bigg(R(a,s,s') + \gamma \sum_{a'} \pi(s',a') Q^\pi(s',a')\Bigg). \quad (15)$$

This is a linear system of equations for Q^π, which can be solved by standard methods.

Why are the action values Q^π so important? With their help, one can improve the present policy unless an optimal policy bringing the biggest reward is found. After all the Q^π have been found, it seems natural to modify the policy π so that, in every state, the action with the largest value is chosen, that is, $a = \arg\max_{a'} Q^\pi(s,a')$. This gives the new policy π_1. It has been proven [14], that $Q^{\pi_1}(a,s) \geq Q^\pi(a,s)$. Then it is possible to do the same thing with $Q^{\pi_1}(a,s)$ and find the next policy π_2 and so on. This process of policy iteration converges to an optimal policy π_* with the corresponding $Q^*(s,a)$. There may be several optimal policies with the same action values. There are very efficient numerical algorithms for policy iteration [14, 83].

The algorithms for Reinforcement Learning (RL) [83, 57] are designed for the case where all the assumptions of the dynamic programming approach are valid, but the transition probabilities P and the rewards R are unknown. This means that we can not find an optimal policy using the equation (15). The task resembles that of the cart-pole control: if we do not know the equations of motion, we can only act by a trial and error method. How could we learn to balance the pole?

The idea of the methods of reinforcement learning is to estimate the same action values Q, but from time averages instead of ensemble averages (15). Hence we need ergodicity. When an agent chooses the action a_t in the state s_t at the moment t and on the next step receives the reward r_{t+1}, the latter can be used for updating the estimate of $Q(s_t, a_t)$. If the number of such trials is large, then the estimates may become accurate enough and may be used for policy improvement. In fact, for improving the present policy one does not need accurate estimates of Q, all that is necessary is to know the a for which $Q(s, a)$ is the largest. This information can often be obtained during a reasonable time.

There are two main classes of methods [83, 57]: Monte-Carlo method in which the Q are estimated as averaged reward received after corresponding state and action (it requires many repetitions), and Temporal Differences method (TD) in which the estimates of these values are received by iterations. Monte Carlo methods usually require many repetitions and converge rather slowly. We will not consider them here. According to [83, 57], TD methods are more popular and usually they converge much faster.

TD methods are based upon the following simple recursive relation: When we are trying to estimate Q experimentally, the estimates for different states are related as $Q(s_t, a_t) = \sum_{k=0}^{\infty} \gamma^k r_{k+t+1} = r_{t+1} + \gamma Q(s_{t+1}, a_{t+1})$, and therefore $\Delta_t = r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, , a_t) = 0$. If $\Delta_t \neq 0$, it means that the present estimates have to be modified. So, the methods look as follows. One uses time-dependent estimates $Q(s, a, t)$ of action values. The initial guesses $Q(s, a, 0)$ can be chosen arbitrarily. Then one chooses a policy π, and after every step modifies the action values iteratively:

$$Q(s, a, t+1) = Q(s, a, t) + \alpha_t e(s, a, t) \Delta_t. \tag{16}$$

This relation contains two new factors — the learning rate α_t and the so-called eligibility traces $e(s, a, t)$, which we shall describe a little later. For the evaluation of the correction term Δ there are two major methods.

(a) "Sarsa" (the name reflects the sequence $s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}$ used for Δ evaluation)

$$\Delta_t = r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}, t) - Q(s_t, a_t, t).$$

It provides the estimates of Q for the current policy. Nonetheless, if one uses the current values of Q for the policy formation, e.g., at the moment t chooses the action a with the largest current $Q(s_t, a, t)$ (usually the choice is more complex, we shall discuss it later), then it proves that the method admits policy improvement as well.

(b) "Q-learning",

$$\Delta_t = r_{t+1} + \gamma \max_a Q(s_{t+1}, a, t) - Q(s_t, a_t, t).$$

This method converges to the action values Q^* for the optimal policy π^* almost independently of the current policy π. Knowing Q^* it is easy to obtain the optimal policy π^* itself.

For the choice of the learning rate α, there is no strong theoretical restrictions. Proof of convergence for the RL methods [83, 57] requires that α_t should slowly decrease with time such that $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$ (requirements of the stochastic approximation theory), for example, $\alpha_t = \alpha_0/t$. Sometimes, the methods work well for nondecreasing α_t that may not be very small. For some problems $\alpha = 1$ is the best choice [57].

Now let us consider eligibility traces. Originally, they appeared in the model of neural learning and were necessary to reproduce effects similar to conditional reflexes [84]. These reflexes relate two events, "neutral" and "essential" that are separated by a time interval. The idea was that neurons and connections responsible for processing neutral events, for some time after them remain eligible for changes. Essential events can cause these changes. As a result, after several repetitions, a neutral event may cause the same effect as an essential one. The corresponding mathematical model of neurons, besides the usual connection weights between neurons w_{ijt}, includes values e_{ijt}, called eligibility traces. The updating rule for the weight takes the form $\Delta w_{ijt} \sim e_{ijt} g_t$, where g_t is a signal generated by an "essential" event. Initially, all $e_{ij0} = 0$. When the neutral event occurs a connection becomes eligible for changes, the corresponding e is set to 1 (or increased by 1). Then it exponentially decays $e_{t+1} = \lambda e_t$, $\lambda < 1$. Therefore, for some time after resetting e, the corresponding connection can change, but later such a possibility vanishes until a new resetting.

In the reinforcement learning, this idea has been implemented as an attempt to solve the mentioned problem of credit assignment. When the correction term Δ is calculated, it would be desirable to update Q not only for the last state s_t, but also for the previous states, since the last reward is partially due to the previous actions. To do this, one has to assign a "credit" factor for those states. In a general situation there is no recipe for this, but for ergodic Markov chains the effect of each action (or being in a specific state) decays exponentially. Therefore, at least for some problems, exponentially decaying eligibility traces may partially solve the problem of credit assignment. The corresponding class of RL algorithms has been called TD(λ), where λ stands for the decay rate of the eligibility traces. For each state-action pair, there is a variable $e(s,a,t)$, $e(s,a,0) = 0$. Then, as soon as the environment is in the state s_t and the action a_t is chosen, the correspondent $Q(s_t, a_t)$ becomes eligible for changes, and we set $e(s_t, a_t, t) = 1$ (or in some versions $e(s_t, a_t, t) = e(s_t, a_t, t) + 1$). For all other state-action pairs eligibility exponentially decay, so finally

$$e(s,a,t) = \begin{cases} 1, & (s,a) = (s_t, a_t), \\ \lambda e(s,a,t-1), & (s,a) \neq (s_t, a_t). \end{cases}$$

The learning may be separated into episodes, for example for the cart-pole problem, after the fall of the pole or hitting the end of the track the cart is returned back in the middle and the pole into the near-vertical position. In such cases eligibility traces should be reset also.

According to [83, 57], eligibility traces may significantly accelerate RL, especially in the case of delayed reward in which nonzero r appears after many actions. As for the choice of the decay rate λ, as well as the discounting parameter γ, there is no theoretical limitations besides the inequalities $0 \leq \lambda < 1$, $0 < \gamma \leq 1$. Some practical recommendations about the choices of α, λ and γ are presented in [90]. But the performance of the methods may essentially depend on their values, so for every specific problem some tests may be necessary. Applications of reinforcement learning are numerous, among them control of complex mechanical systems, navigation of robots, elevator dispatching, playing backgammon, optimal chaos control and others, see [83, 57, 92, 36, 97].

Now, how do all these relate to neural networks and complex dynamics? Relations with neural networks are numerous. First, the methods of reinforcement learning involve a number of functions, approximation of which is required for some versions of RL [57, 83]: $\pi : s \to a$, $Q : s \times a \to R^1$,

$R, F : s\times s\times a \to R^1$. Such an approximation can be done with the help of one of the approximating networks, e.g., a multilayer perceptron. Second, often the state s changes continuously. The corresponding methods of dynamic programming and reinforcement learning exist, but they are more complex than those for discrete cases. For this reason the space of possible states is subdivided into a number of domains, and each domain is considered as a single discrete state. This procedure of state space quantizing can be naturally done with the help of vector quantizing neural networks, such as Kohonen's or ART [18, 59]. For example, a neural controller may look like a two-stage Kohonen's network. It receives s as its input, each neuron corresponds to a discrete state. After the competition, the neuron corresponding to the present state turns on and sends a signal to the second network through a number of output connections with the weights equal to Q. In the second winner-takes-all network each neuron corresponds to a separate action. After a competition an action with the largest Q is selected. Therefore, neural networks are good tools for RL methods, sometimes even the term "neurodynamic programming" is used [15].

Relation with complex dynamics and chaos also exists. First, as it has been mentioned in the previous section, a neural controller with reinforcement learning integrated with an unstable dynamical system can learn to stabilize it in a chaotic state. Fig. 4 shows the example of learning for the cart-pole problem. Second, chaos may be necessary for the controllers implementing RL methods for obtaining a good policy with exploration. Let us consider the problem in more details.

6.2. Reinforcement Learning, Exploration, and Chaos

Let us now discuss policy making, that is, choosing criteria for taking action at a given state. In the dynamic programming, when information about the environment is complete, the action values Q^* for optimal policy can be calculated, the optimal way is to use actions with the largest Q^* . But, in learning by a trial and error method, this approach may not be optimal, because only estimates and not the true values of Q^* are available. Even if some of the Q are estimated accurately, there always remains a possibility that the best action might not have been tried. This means that the true Q^* may still remain unknown. If an agent always uses only the best known actions, then it may never learn the optimal policy. Hence, at times, it is necessary to try some

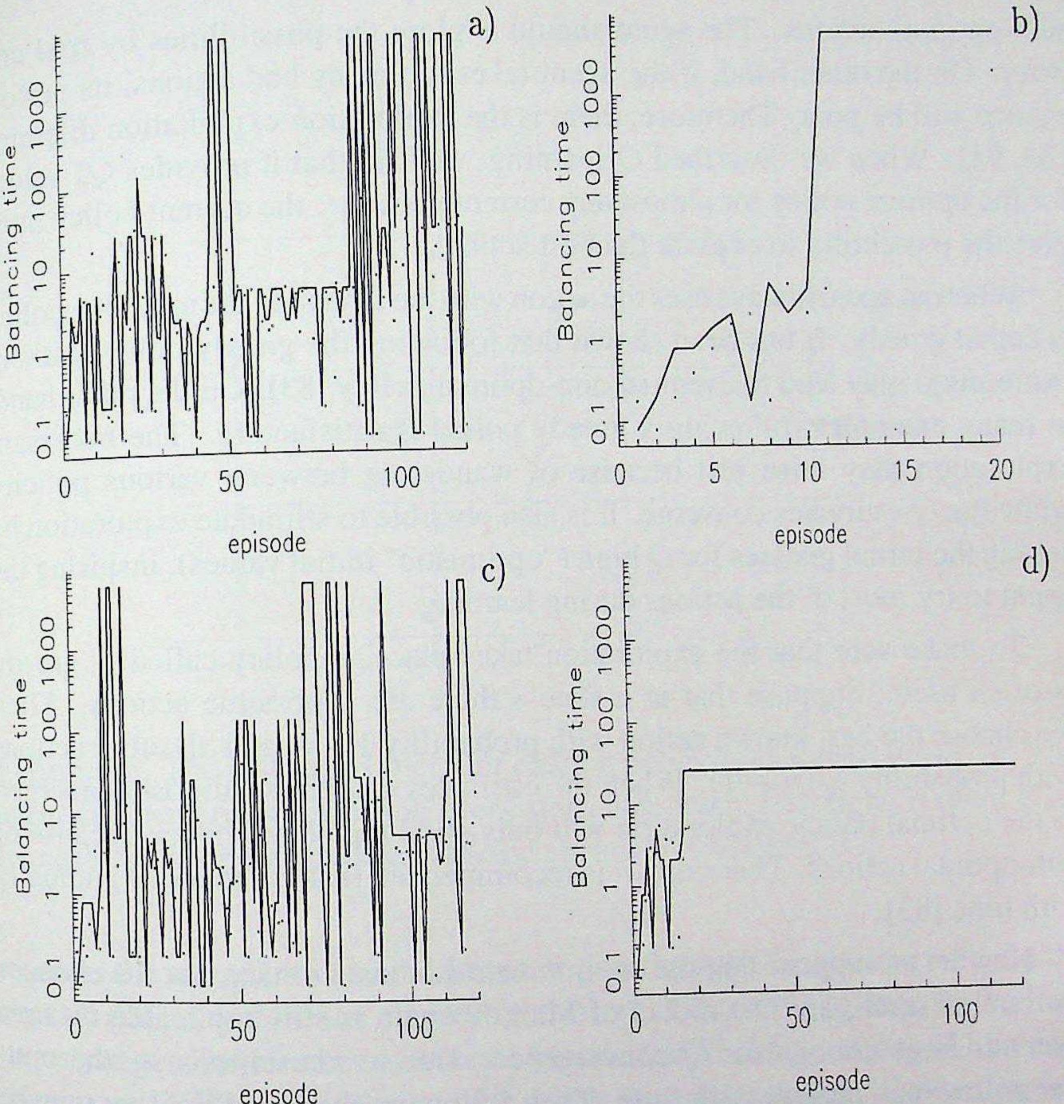

Figure 4. Examples of reinforcement learning (Q-learning) for the cart-pole control. The learning is subdivided into episodes. Episode ends either if x or θ fall outside admissible limits (then the controller receives negative reinforcement signal) or if control is successful until $T = 6000$. The latter means that the controller can keep the pole for a very long (probably infinite) time. The duration of each episode is shown by dots. During learning the policy may be ϵ-greedy (panels a and c), and failure may be only due to random action. After each episode the learned greedy policy also was tested, and the duration of the tested episode is shown by solid line. Panels correspond to different values of the parameters of Q-learning. For the good choice (panel b) learning is efficient. Panels a and c show, that for some parameter values the learned policy may oscillate: controller finds a good policy, then forgets it, then finds again.

non-optimal actions. The agent should explore the possibilities by trial and error. On the other hand, if the agent takes too many bad actions, its performance will be poor. Therefore, there is the exploration-exploitation dilemma [83, 91]. When we described Q-learning, we said that it provides Q^* values for the optimal policy for almost any current policy π: the current policy must give the possibility to explore the best actions.

When an agent always uses the action with the largest Q estimate, the policy is called greedy. It has been shown that following the greedy policy without exploration may lead to eventual non-optimal policy [83]. On the other hand, in many examples following a greedy policy is satisfactory. The necessary exploration may arise just because of wandering between various policies while the Q estimates converge. It is also possible to stimulate exploration by setting the initial guesses for Q high ("optimistic" initial values), inspiring the agent to try most of the actions during learning.

To make sure that the exploration takes place, a policy called ϵ- greedy is often used. Suppose that in a state s there are n possible actions. Then we choose the best known action with probability $1 - \epsilon$, and all other actions with probability $\epsilon/(n-1)$. When the estimates of Q have almost converged to the optimal policy, exploration will only spoil the performance because of non-optimal actions. Therefore, it is recommended that ϵ is reduced gradually with time [83].

Now let us suppose that the environment is nonstationary, but the changes are rare or small, and the model of Markov chain is still applicable on time intervals large enough for Q convergence. Due to nonstationarity the optimal policy may change with time. Then following the greedy policy may be dangerous: there is no more exploration due to initial values or policy convergence, and therefore a new optimal policy may never be found. Under such circumstances special exploration, e.g., in the form of an ϵ-greedy policy may become important. This can be shown with an example of navigation in a gridworld, a typical test problem for the reinforcement learning [57, 83].

In the example of navigation in a gridworld, the environment consists of a number of cells, and the agent can move from a cell to one of its neighbors, Fig. 5.a. The possible actions are up, down, left, and right. If there is no neighboring cell in a chosen direction or if this cell is blocked (dashed cells in Fig. 5.a), then the agent stays where it is. The agent begins its motion from the starting cell marked "S", and receives a reward $r = -1$ after every action unless it reaches the goal cell "G." If the agent reaches the cell G, it

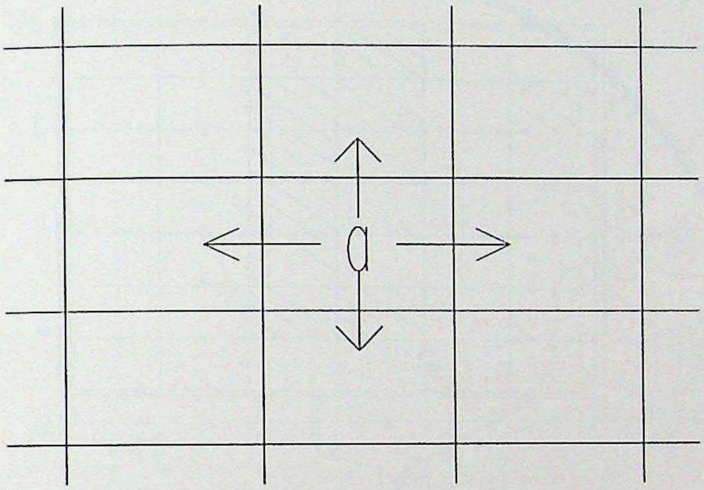

Figure 5.a. The testing problem for chaotic policy—a shortcut maze. (a) The agent can perform 4 actions in every state: up, down, left, right, if there is no free cell in the specified direction, the agent stays where it is.

receives the reward $r = 1$, and instantly goes back to the starting cell. So the agent's goal is to find the shortest path from "S" to "G" to receive the least punishment. The number of states s here is equal to the number of free cells minus one ("G"), and in every state 4 actions are available. Events that take place in the journey from "S" to "G" form an episode. The duration of the current episode T may be considered as a measure of the quality of the learned policy. We characterize the current ϵ-greedy policy by the averaged ratio of minimal possible duration $T_{\min}$ and T: $q = \langle T_{\min}/T \rangle$, the averaging is done over 100 different runs with the same ϵ. For the choice of actions we numbered them, and sometimes the results depended on the order. For this reason the results are presented for all 24 possible orderings of four actions.

If the environment remains unchanged, no special exploration is necessary, both sarsa and Q-learning with greedy policy find optimal or almost optimal policy. Exploration can only spoil the performance, Fig. 6.da, so we used a nonstationary environment [76]. After 2010 steps, the environment changes: the uppermost closed (dashed) cell in Fig. 5.ab opens, Fig. 5.ac, and in the

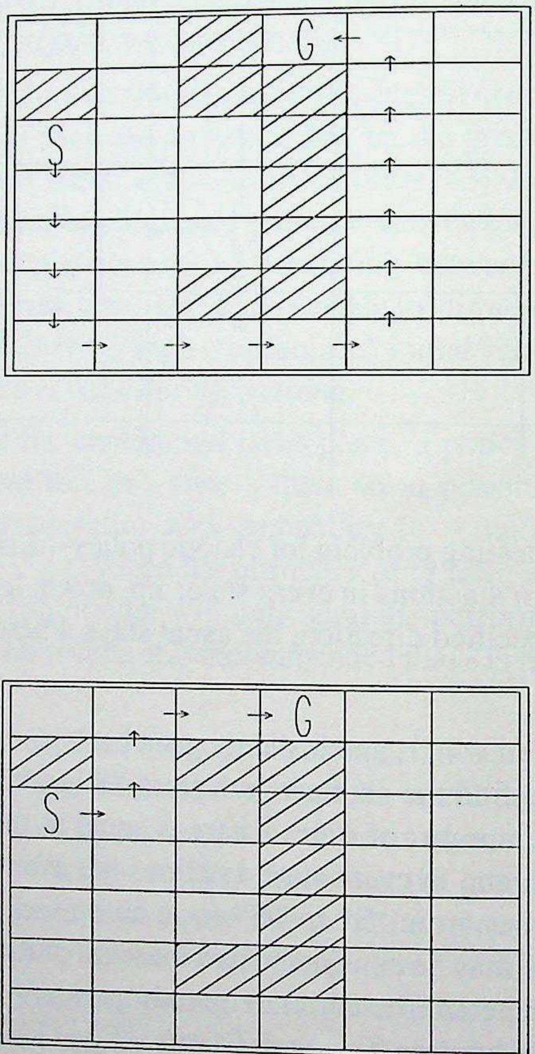

Figure 5.b,c. The testing problem for chaotic policy—a shortcut maze. (b) Original environment, the agent starts from the "S" cell and should reach the goal "G." The shaded cells are not available for the agent, and the shortest way takes 15 steps. At every step it receives the reward $r = -1$, unless the goal is reached; then it receives the reward $r = +1$ and returns to the starting point. (c) After $t = 2010$ one cell opens, and the shorter 5-steps way appear. Exploration is necessary to "discover" this short path.

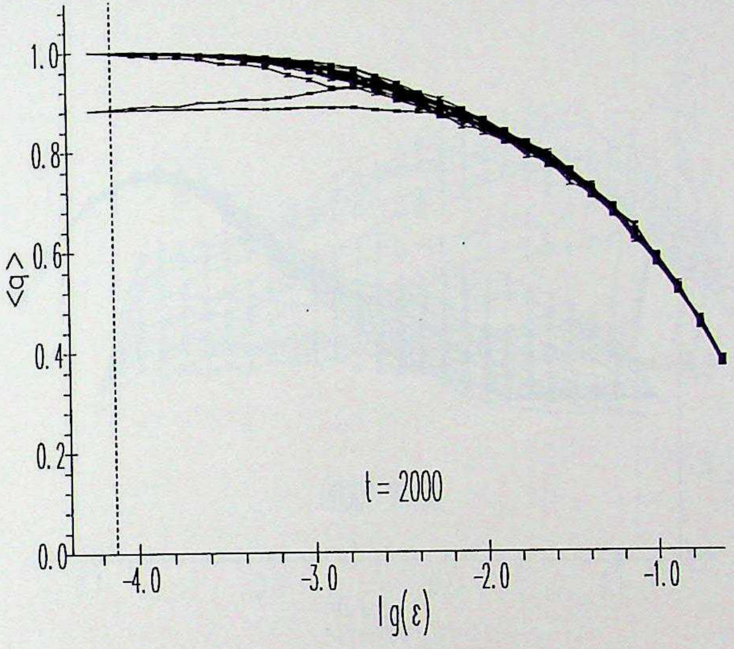

Figure 6.a. Results for exploration tests in Fig. 5.a. Different curves correspond to various numberings of the four actions. Error marks show the error estimate after averaging over 1000 runs. The leftmost point, separated by dotted vertical line, corresponds to $\epsilon = 0$. (a) For stationary environment random exploration only spoil the performance: the greater ϵ the less is q.

new environment the shortest path takes only 5 steps instead of 15. Adaptation to the new environment can not be achieved without special exploration in the algorithms. Now there is an optimal degree of exploration, plots in Fig, 6.db has a maximum.

Exploration requires randomness in the algorithm. A learning system must possess some source of the random signal. And here we come to the natural place of dynamical chaos in the process of learning. Chaos may be a source of random-like decisions in policies with exploration. To test this hypothesis we repeated the previous experiment with random, regular and chaotic policies. To choose actions for ϵ-greedy policy, we used the sequence of pseudorandom numbers as well as numbers generated by nonchaotic and chaotic dynamical systems. Fig. 6.dc,d. For a chaotic sequence without noticeable correlations (every k-th number or data from k independent systems) the results were as good as those with random numbers.

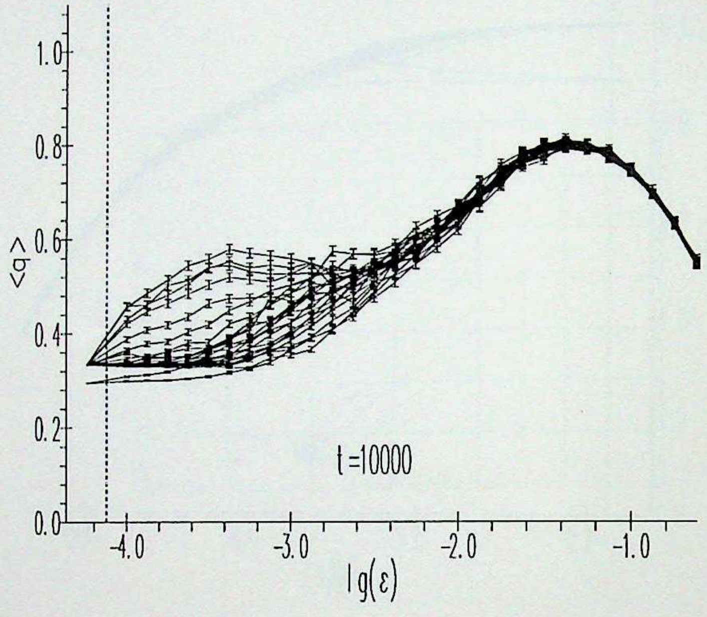

Figure 6.b. After change in the environment exploration may increase average performance: there is an optimal noice level for which performance is the best.

Among chaotic systems, the best results were obtained for the robust chaotic neuron, the mapping $x_{t+1} = |\tanh s(x_t - c)|$ [11, 75]. Therefore, neural networks can both be used for implementing the methods of reinforcement learning and at the same time be a source of exploration.

7. Hamiltonian Neural Networks: Pattern Recognition without Dissipation

Recently, we have seen a feverish activity in the field of quantum computing and quantum information processing, see e.g., [56, 68, 98, 94] for a review. There is also a possibility that quantum mechanisms are used by the brain [71]. At the moment, it is hard to predict what quantum devices will appear in the future. However, one thing is sure that quantum computers are and will remain Hamiltonian systems. The importance of quantum-classical correspondence has been realized since the very beginning of quantum mechanics. Although,

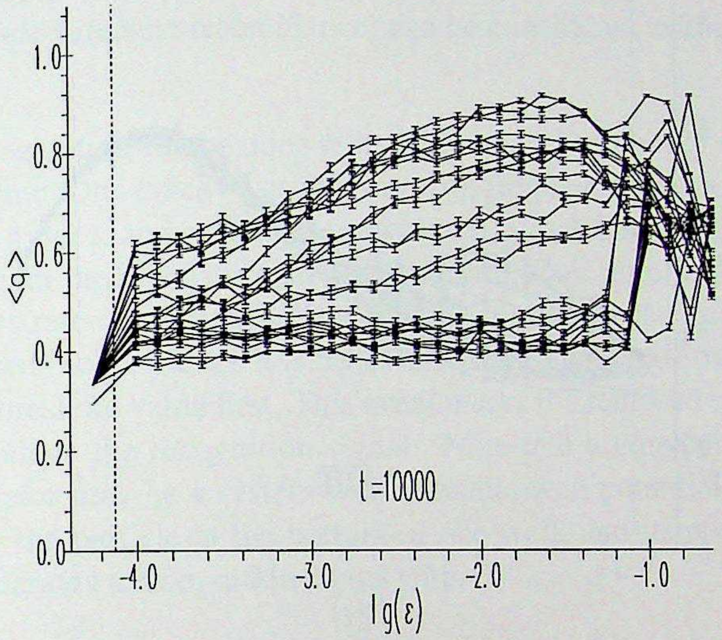

Figure 6.c. Nonchaotic exploration, a trajectory of logistic map at a_∞ is used for action selection, regularity leads to strong dependence of the results on numbering scheme.

there are difficulties in the pursuit of this correspondence, the quest for better understandings of this correspondence will remain valid and desirable as long as we have quantum and classical mechanics as separate entities for describing natural phenomena. None of the neural networks that we have described above qualifies for quantum analysis since they are not Hamiltonian systems. This has led us to look into Hamiltonian neural networks.

Hamiltonian neural networks (HNNs) can be linear, nonlinear, chaotic, regular, autonomous or non-autonomous. Obviously, the field is wide. The simplest HNN is a linear autonomous Hamiltonian dynamical system. This is a conservative system, nothing comes in or goes out of the system. In contrast, the neural networks that we have discussed in previous sections are dissipative. In traditional neural networks, dissipation plays an important role. It works as a filter to eliminate unnecessary (noisy) information from the input data in pattern recognition. The very existence of attractors is based on dissipation. Thus, for pattern recognition with autonomous HNN, we need to resort to

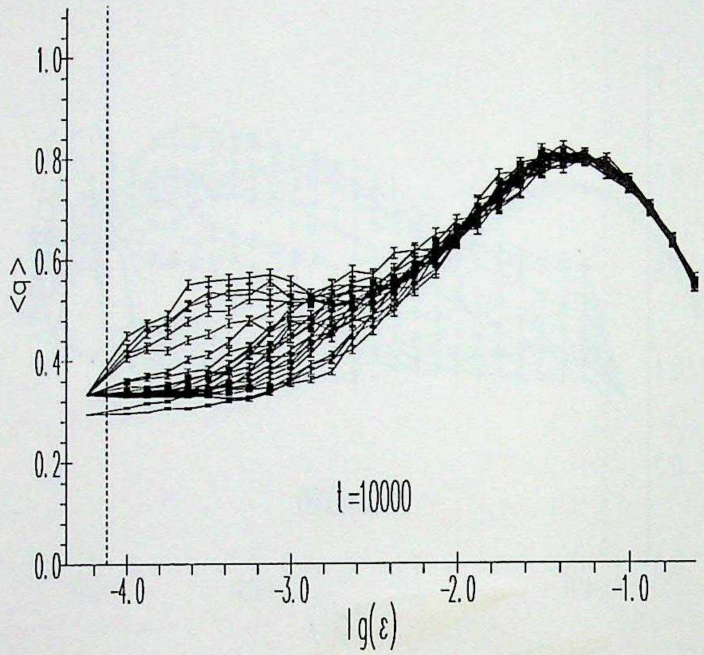

Figure 6.d. Chaotic exploration, every 8-th point of the robust chaotic heuron. The results are as good as for random exploration. Therefore, chaos may be used for exploration purposes.

techniques that are different from the traditional ones.

In this section we would like to describe a number of ideas, which can be used for the development of Hamiltonian neural networks. We shall describe an implementation of a subspace projection technique and a winner-take-all architecture. Together they can make a Hamiltonian pattern recognition network.

A linear Hamiltonian dynamical system with oscillatory behavior can be written in a form $\dot{z} = (U + iW)z$, where U is an antisymmetric and W a symmetric real matrix. Here we use $z = x + ip$ as a complex N-dimensional vector. For simplicity, let us set $U = 0$. Then this system is equivalent to N oscillators with the frequencies ω_k—eigenvalues of W. Each oscillatory mode corresponds to the eigenvector e_k of W. These oscillatory modes are called "normal modes." When we present the initial data z_0, they are decomposed into a set of those modes, $z(t) = \sum_k (z_0, e_k)e^{i\omega_k t}$, the amplitude of each mode being proportional to the projection of the input pattern onto e_k.

If we can construct a matrix W such that the e_k are the stored patterns, then the amplitudes of modes correspond to the degree of resemblance of the input pattern with each of the stored patterns. Therefore, the mode with the largest amplitude (greatest resemblance) can be considered as the recognized pattern.

The mode selection can be done with the help of the effect of resonance in excited oscillator. One takes N oscillators with frequencies ω_k equal to those of the normal modes, and excites them with the signal from the Hamiltonian system. Since all the oscillators are in resonance, their amplitude will grow, and the largest rate of growth will be for the oscillator that corresponds to the most excited mode. Hence it is necessary to find which of the amplitudes passes some threshold value first. This event marks the required oscillator and hence can produce the recognition signal. Note that a convenient form for such an oscillator may be a system with a double-well potential. Initial data correspond to the particle at the bottom of one well, and its presence in the other well generates a recognition signal (Fig. 7).

The only remaining problem is to find out how to store a set of generically non-orthogonal patterns into orthogonal normal modes without distorting their form. This problem can be solved with the help of hidden dimensions.

Suppose that the vectors $\xi_1, \ldots, \xi_M \in R^{n_1}$, representing the patterns to be stored, are not orthogonal but linearly independent. We want to make them orthogonal without changing the patterns themselves. Let us show that it can be done by adding $L \geq M$ new dimensions. The new components of the extended vectors are denoted by v_k. It is convenient to consider them as vectors in R^L. So the task is to make new orthogonal vectors $\zeta_k = \{\xi_k, v_k\} \in R^n$, $n = n_1 + L$. The vectors v_k should satisfy the equations

$$(\xi_k \cdot \xi_j)_{n_1} + (v_k \cdot v_j)_L = A_k \delta_{kj}, \quad k, j = 1, \ldots, M. \tag{17}$$

Here $(\cdot)_{n_1}$ and $(\cdot)_L$ denote dot products in the subspaces of ξ and v respectively. Let us show that for A_k sufficiently large the solution for this problem exists.

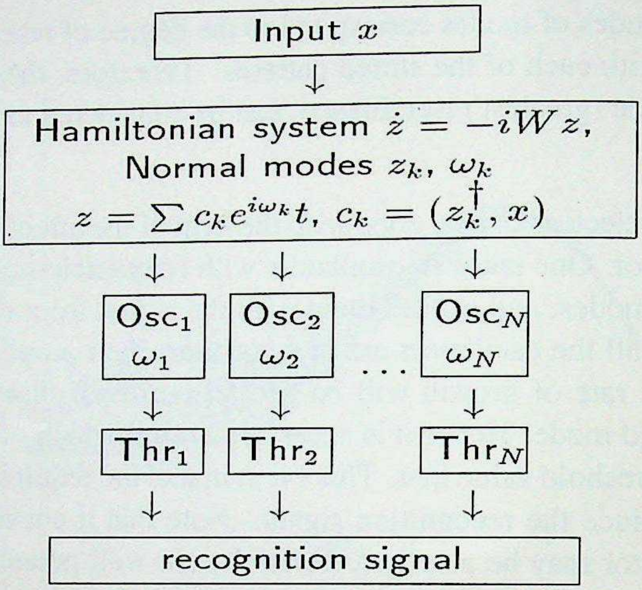
Input x
Hamiltonian system $\dot{z} = -iWz$,
Normal modes z_k, ω_k
$z = \sum c_k e^{i\omega_k t}$, $c_k = (z_k^\dagger, x)$
Osc$_1$ ω_1
Osc$_2$ ω_2
Osc$_N$ ω_N
Thr$_1$
Thr$_2$
Thr$_N$
recognition signal

Figure 7. Scheme of Hamiltonian neural network. Input x is used as initial data for the Linear Hamiltonian system. The latter decomposes it into oscillatory modes z_k. The modes excite oscillators Osc with the frequencies ω_k equal to that of the modes. Amplitude of the oscillators grow with the rate $\sim |c_k|$ due to resonance. Threshold detectors Thr determine, which of the amplitudes overcome a threshold value first and generate a recognition signal, defining the winning mode. Learning of this system is related with calculating of the matrix W.

It is convenient to consider the vectors v_k as columns of an $L \times M$ matrix V. Let the latter has the form

$$V = \begin{pmatrix} d & v_{12} & v_{13} & v_{14} & & v_{1M} \\ & d & v_{23} & v_{24} & & v_{2M} \\ & & d & v_{34} & & \vdots \\ & & & d & & \vdots \\ & & 0 & & \ddots & v_{M-1,M} \\ & & & & & d \end{pmatrix}.$$

Then the conditions (17) for $k = 1$ take the form

$$(\xi_1 \cdot \xi_j) + d \cdot v_{1j} = A_1 \delta_{1j},$$

therefore

$$v_{1j} = -(\xi_1 \cdot \xi_j)/d, \quad j = 2, \ldots, M,$$

$A_1 = |\xi_1|^2 + d^2$. So, we have chosen the first column and the first row of V and have made the vector $\zeta_1 = \{\xi_1, v_1\}$ orthogonal to all other ζ_k, irrespective of the choices of v_k. Then we make the rest of the vectors orthogonal to ζ_2:

$$(\xi_2 \cdot \xi_j) + v_{12}v_{1j} + d \cdot v_{2j} = A_2\delta_{2j},$$

and

$$v_{2j} = -\frac{(\xi_2 \cdot \xi_j) + v_{12}v_{1j}}{d}, \quad j = 2, \ldots, M,$$

$A_2 = |\xi_2|^2 + v_{12}^2 + d^2$. After repeating this procedure $M - 1$ times we find all the components of v_k. This proves the orthogonality of the resulting extended vectors ζ_k.

Test calculations show that this scheme of pattern recognition works and one can store up to N linearly independent images and correctly recognize them. The presented pattern recognition scheme can be considered as a new implementation of a subspace classification technique [59].

8. Conclusions and Outlook

We have presented an overview of dynamical neural networks that perform information processing. There is an input X data and an output $Y = \varphi^t(X)$, where φ is a solution of differential or difference equations of the network dynamics. Except for the tasks of modeling a time series [45], one is interested in invariant sets of φ, that is, attractors. Many different techniques can bring up the same result, therefore one looks for the most efficient one. Efficiency, however, is a relative attribute, it depends on currently available tools of operation. Nonetheless, we must say that the scheme, initial data → attractor (or parameters → attractor) is not a universal one, since information can come gradually in time. For example, as they say, "if it quacks like a duck (time evolution), walks like a duck (time evolution), and has web feet (time invariant property), then it is the duck." Sometimes such tasks can be transformed into usual ones by recording temporal process and then presenting it as a high-dimensional vector, but usually similar problems now are treated by Artificial Intelligence techniques rather than neural networks. One asks: how can the area of applicability of neural networks be extended and chaos made useful in their performance?

We considered a number of successful applications and interesting approaches to possible future applications of dynamical chaos in neural networks in Section 4. However, they do not extend the area of applicability of existing neural networks. If one is interested in a definite answer (time independent Y) from a network, what can be the role of chaos that gives unpredictable answers? From the point of view of a definite final answer, an isolated chaotic network is not very useful, except for the role of a specific "lubricant" which prevents sticking to a wrong attractor.

However, if a neural network forms a part of a global system, there can be substantial benefit. Let us call a network an 'embodied network' if it forms a controlling part of a complex autonomous system. Embodied neural networks perform new types of tasks that cannot be done by isolated ones: an embodied network can learn through the interaction with the 'world'. An example is reinforcement learning considered in Sect. 6. For these tasks, sometimes it does not matter whether the output of an isolated network is predictable or unpredictable. Often an autonomous agent does not have an exactly specified goal; it simply has some criteria for success. As we have shown in the text, there is at least one possible niche for chaos in the interactive learning. Maybe others also exist.

We can conclude that embodied neural networks are a new and interesting field of studies from the viewpoint of nonlinear dynamics. The only problem is that one needs a controlled system to study an embodied network. Chaotic systems such as logistic map, Lorenz system, and Henon map have helped formulate most of the concepts of contemporary nonlinear dynamics. We hope that a number of "standard" subsystems and control tasks for studies of embodied networks will appear in the future. Most probably such studies will be related with robotics and the direction called "Artificial Life" (see, e.g., [73] for review). Small autonomous robots or "creatures" controlled by simple neural networks have been modeled and built for a number of years. Typically, these artificial autonomous creatures do not learn. A necessary configuration of neural network arises with the help of evolutionary or genetic algorithms. The resulting behavior of the creatures often looks chaotic. However, to our knowledge, such behaviors have not been studied in detail. Now there is a challenge of creating creatures which could both evolve and learn. And it is likely that dynamical chaos will be a typical state for neural networks needed for such tasks [13], or that such networks will need a chaotic subsystems. Embodied neural networks and artificial life seem to be a natural home for nonlinear dynamics and chaos.

Dynamical systems cover the vast field of natural phenomena, while currently known neural networks fill only a small part of it. We have microscopic, mesoscopic and macroscopic systems. The concept of information processing by neural networks will extend to areas other than what is known at the present time. An immediate extension is expected to happen in quantum computation that is based on Hamiltonian neural networks. In the text, we have presented a simple example of classical Hamiltonian neural network. Currently, very little is known about Hamiltonian neural networks. We anticipate more activities in this direction.

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Stability and Bifurcation in a Neural Network Model with Delay

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Abstract

In this paper we investigate the existence of periodic solutions of a simple neural network model with delay. At first, linearized stability and local Hopf bifurcation are studied. Secondly, the stability and direction of the Hopf bifurcation are determined by applying the normal form theory and center manifold theorem. Finally, the existence of the global Hopf bifurcation is investigated by using the method in [3].

1. Introduction

In the last several decades, a great deal of research has been devoted to the neural network. However, time delays always occur in the signal transmission among neurons due to the finite propagation velocity of the electrical signals along axons, the finite speed of neurontransmitters across the synaptic gap and the finite switching speed of amplifiers(neurons) in the electronic hardware implementation. Hence, many researchers pay more attention to delay-induced changes in the dynamics of neural networks (see[1], [2], [3], [4], [5] and the references there in).

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In this paper, we consider the network which consists of an excitatory neuron and an inhibitory neuron. In [7], the mathematical model describing the dynamics of this kind of neural network was given by

$$\begin{cases} \frac{du_1}{dt} = & -u_1 + q_{11}\varphi(u_1) - q_{12}u_2 + e_1, \\ \frac{du_2}{dt} = & -u_2 + q_{21}\varphi(u_1) + e_2. \end{cases} \tag{1.1}$$

The functions and parameters occurring in system (1.1) have the following neurophysiological meaning:

- $u_1, u_2 : \mathbb{R} \longrightarrow \mathbb{R}$ denote the total potential of the excitatory and inhibitory neuron, respectively.

- q_{ij} is a positive constant that represent the strength of the connection line from the jth neuron to the ith neuron. In this work we assume $q_{22} = 0$, i.e., there is no self-inhibition in the 2nd neuron.

- $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^+$ is the transfer function that describes the activity generation of the 1th (excitatory) neuron as a function of its total $u_1(t)$. In this work, we choose $\varphi(u_1) = \frac{1}{1+\exp(-4u_1)}$. For the sake of simplicity, the transfer function corresponding to the inhibitory neuron is assumed to be the identity.

- e_1, e_2 are external stimuli acting on the 1st (excitatory) and 2nd (inhibitory) neuron, respectively.

In [6], the bifurcation properties of the system (1.1) for fixed q_{11} were studied. In [7], the periodic orbits of the system (1.1) were studied. However, time delays always occur in the signal transmission between neurons. So, the following system (1.2) with delay describes the dynamics of the neuron network more exactly than the system (1.1) does.

$$\begin{cases} \frac{du_1}{dt} = -u_1 + q_{11}\varphi(u_1) - q_{12}u_2(t-\tau) + e_1, \\ \frac{du_2}{dt} = -u_2 + q_{21}\varphi(u_1(t-\tau)) + e_2. \end{cases} \tag{1.2}$$

Setting

$$u = u_1, \text{ and} v = \frac{1}{q_{21}}(u_2 - e_2).$$

The system (1.2) becomes

$$\begin{cases} \frac{du}{dt} = -u + a\varphi(u) - bv(t-\tau) + c, \\ \frac{dv}{dt} = -v + \varphi(u(t-\tau)), \end{cases} \tag{1.3}$$

where $a = q_{11} > 0, b = q_{12}q_{21} > 0, c = e_1 - q_{12}e_2$. The physical meaning of the new parameters a, b and c is clear: a describes the positive feedback, b describes the negative feedback of the system, and c describes the total external input acting on the 1st neuron.

In the following section, $\tau \in \mathbb{R}$ is set to be a parameter to study the stability and Hopf bifurcation of the system (1.3).

2. Stability of the Equilibria and Local Hopf Bifurcation Analysis

It can be easily verified that $(u, v) \in \mathbb{R}^2$ is an equilibrium of (1.3) if and only if

$$\begin{cases} c = u - (a - b)\varphi(u), \\ v = \varphi(u). \end{cases} \tag{2.1}$$

Lemma 2.1 ([7]). (Existence of equilibria)

1. If $1 \geq a - b$, then there is a unique equilibrium of (1.3) for each $c \in \mathbb{R}$.
2. If $1 < a - b$, then there are $c_1, c_2 \in \mathbb{R}$ with $c_2 < c_1$ such that
   - (a) for $c \in (-\infty, c_2) \cup (c_1, +\infty)$ there exists exactly one equilibrium,
   - (b) for $c \in \{c_1, c_2\}$ there exist exactly two equilibria,
   - (c) for $c \in (c_2, c_1)$ there exist exactly three equilibria.

It follows from Lemma 2.1 that (1.3) admits at least one equilibrium for every choice of a, b, c. Let $(u^*, v^*) \in \mathbb{R}^2$ be an equilibrium of (1.3). Using new coordinates

$$x = u - u^*, y = v - v^*.$$

The system (1.3) becomes

$$\begin{cases} \frac{dx}{dt} = -x + a\varphi(x + u^*) - by(t - \tau) - u^* - bv^* + c, \\ \frac{dy}{dt} = -y + \varphi(x(t - \tau) + u^*) - v^*. \end{cases} \tag{2.2}$$

The linearization of system (2.2) at the equilibrium (0, 0) is

$$\begin{cases} \frac{dx}{dt} = (a\varphi'(u^*) - 1)x - by(t - \tau), \\ \frac{dy}{dt} = -y + \varphi'(u^*)x(t - \tau). \end{cases} \tag{2.3}$$

The associated characteristic equation of (2.3) is

$$\lambda^2 + (2 - a\varphi'(u^*))\lambda + 1 - a\varphi'(u^*) + b\varphi'(u^*)e^{-2\lambda\tau} = 0. \tag{2.4}$$

Lemma 2.2. If

$$(H_1) : \max\{0, 1 - b\varphi'(u^*)\} < a\varphi'(u^*) < \min\{2, 1 + b\varphi'(u^*)\}$$

holds, then Eq. (2.4) with $\tau = \tau_n$ has a pair of simple purely imaginary roots $\pm i\omega_0$;

If

$$(H_2) : 0 < a\varphi'(u^*) < 1 - b\varphi'(u^*) \text{ or } a\varphi'(u^*) > 1 + b\varphi'(u^*)$$

holds, then Eq. (2.4) has no pure imaginary roots, where

$$\begin{aligned} \omega_0 \;=\; & \frac{1}{\sqrt{2}}\Big[(((a\varphi'(u^*) - 1)^2 - 1)^2 + 4b^2\varphi'^2(u^*))^{\frac{1}{2}} \\ & -((a\varphi'(u^*) - 1)^2 + 1)\Big]^{\frac{1}{2}}, \end{aligned} \tag{2.5}$$

$$\tau_n = \frac{1}{2\omega_0}(\sin^{-1}\frac{(2 - a\varphi'(u^*))\omega_0}{b\varphi'(u^*)} + 2n\pi)(n = 0, 1, 2, \ldots), \tag{2.6}$$

and

$$0 < \sin^{-1}\frac{(2 - a\varphi'(u^*))\omega_0}{b\varphi'(u^*)} < \pi.$$

Proof. $\lambda = i\omega(\omega > 0)$ is a purely imaginary root of Eq. (2.4) if and only if ω satisfies

$$\begin{cases} \omega^2 + a\varphi'(u^*) - 1 = b\varphi'(u^*)\cos 2\omega\tau, \\ (2 - a\varphi'(u^*))\omega = b\varphi'(u^*)\sin 2\omega\tau. \end{cases} \tag{2.7}$$

It follows from (2.7) that

$$\omega^4 + ((a\varphi'(u^*) - 1)^2 + 1)\omega^2 + ((a+b)\varphi'(u^*) - 1)((a-b)\varphi'(u^*) - 1) = 0. \tag{2.8}$$

Setting $z = \omega^2$, (2.8) becomes

$$z^2 + ((a\varphi'(u^*) - 1)^2 + 1)z^2 + ((a+b)\varphi'(u^*) - 1)((a-b)\varphi'(u^*) - 1) = 0. \tag{2.9}$$

If (H_1) holds, it is easy to verify that (2.9) has a positive real root and a negative real root. But (2.9) has no positive real root when (H_2) holds. By (H_1) , (2.9) has a positive root

$$z_0 = \frac{1}{2}((((a\varphi'(u^*) - 1)^2 - 1)^2 + 4b^2\varphi'^2(u^*))^{\frac{1}{2}} - ((a\varphi'(u^*) - 1)^2 + 1)).$$

Hence, $\omega_0 = z_0^{\frac{1}{2}}$ is the only positive real root of Eq. (2.8). Let

$$\tau_n = \frac{1}{2\omega_0}\Big[\sin^{-1}\frac{(2 - a\varphi'(u^*))\omega_0}{b\varphi'(u^*)} + 2n\pi\Big], (n = 0, 1, 2, \dots.)$$

where $0 < \sin^{-1}\frac{(2-a\varphi'(u^*))\omega_0}{b\varphi'(u^*)} < \pi$ and its quadrant is decided by the sign of $\omega_0^2 + a\varphi'(u^*) - 1$.

From (2.7) we know that Eq. (2.4) with $\tau = \tau_n, n = 0, 1, 2, \dots$ has a pair of purely imaginary roots $\pm i\omega_0$. In fact, this pair of roots $\pm i\omega_0$ are also simple. Otherwise, we have

$$\begin{cases} \lambda^2 + (2 - a\varphi'(u^*))\lambda + 1 - a\varphi'(u^*) + b\varphi'(u^*)e^{-2\lambda\tau}|_{\tau=\tau_n} = 0, \\ 2\lambda + (2 - a\varphi'(u^*)) - 2\tau b\varphi'(u^*)e^{-2\lambda\tau}|_{\tau=\tau_n} = 0. \end{cases} \tag{2.10}$$

It follows from (2.10) that

$$2\tau\lambda^2 + (2 + 4\tau - 2\tau a\varphi'(u^*))\lambda + 2\tau(1 - a\varphi'(u^*)) + 2 - a\varphi'(u^*) = 0. \tag{2.11}$$

By $a\varphi'(u^*) < 2$, we get

$$2 + 4\tau - 2\tau a\varphi'(u^*) = 2 + 2\tau(2 - a\varphi'(u^*)) > 0.$$

This implies that (2.10) can not have a pair of purely imaginary roots $\pm i\omega_0$ for $\tau = \tau_n$, which contradicts the above discussion. Hence, we obtain that $\pm i\omega_0$ is a pair of purely imaginary simple roots of Eq. (2.4). This completes the proof.

Let

$$\lambda_n(\tau) = \eta_n(\tau) + i\omega_n(\tau)$$

be the root of Eq. (2.4) with $\eta_n(\tau_n) = 0$, $\omega_n(\tau_n) = \omega_0$. Then we have the following result.

Lemma 2.3. The following transversality condition holds:

$$\frac{dRe\lambda_n(\tau)}{d\tau}|_{\tau=\tau_n} > 0.$$

Proof. From Eq. (2.4), we have

$$2\lambda_n(\tau)\frac{d\lambda_n}{d\tau}+(2-a\varphi'(u^*))\frac{d\lambda_n}{d\tau}-2b\varphi'(u^*)\lambda_n e^{-2\lambda_n\tau}-2\tau b\varphi'(u^*)e^{-2\lambda_n\tau}=0,$$

that is

$$\frac{d\lambda_n(\tau)}{d\tau}=\frac{2b\lambda_n\varphi'(u^*)e^{-2\tau\lambda_n}}{2\lambda_n+2-a\varphi'(u^*)-2\tau b\varphi'(u^*)e^{-2\tau\lambda_n}}.$$

Substituting $\tau=\tau_n$ into the above equation, we obtain

$$\frac{d\lambda_n(\tau_n)}{d\tau}=\frac{2\omega_0(b\varphi'(u^*)\sin 2\omega_0\tau_n+ib\varphi'(u^*)\cos 2\omega_0\tau_n)}{(2-a\varphi'(u^*)-2\tau_n b\varphi'(u^*)\cos 2\omega_0\tau_n)+(2\omega_0+2\tau_n b\varphi'(u^*)\sin 2\omega_0\tau_n)i}.$$

Hence,

$$\begin{aligned}\frac{dRe\lambda_n(\tau_n)}{d\tau} &= \frac{2\omega_0}{\triangle}\{b\varphi'(u^*)(2-a\varphi'(u^*)-2\tau_n b\varphi'(u^*)\cos 2\omega_0\tau_n)\sin 2\omega_0\tau_n\\ &\quad +b\varphi'(u^*)(2\omega_0+2\tau_n b\varphi'(u^*)\sin 2\omega_0\tau_n)\cos 2\omega_0\tau_n\}\\ &= \frac{2\omega_0^2}{\triangle}((a\varphi'(u^*)-1)^2+2\omega_0^2+1)>0,\end{aligned}$$

where

$$\triangle=(2-a\varphi'(u^*)-2\tau_n b\varphi'(u^*)\cos 2\omega_0\tau_n)^2+(2\omega_0+2\tau_n b\varphi'(u^*)\sin 2\omega_0\tau_n)^2.$$

Lemma 2.4. Suppose (H_2) is satisfied. Then all roots of Eq. (2.4) have strictly negative real parts for $\tau\geq 0$; Suppose (H_1) is satisfied. We have

(i) If $\tau\in[0,\tau_0)$, then all roots of Eq. (2.4) have strictly negative real parts;

(ii) If $\tau=\tau_0$, then Eq. (2.4) has a pair of purely imaginary roots $\pm i\omega_0$, which are simple, and all other roots have strictly negative real parts;

(iii) If $\tau>\tau_n(n=0,1,2,\ldots)$, then Eq. (2.4) has at least $n+2$ roots with strictly positive real parts.

Proof. From the above discussion, we know that τ_0 is the first value of τ at which Eq. (2.4) has a pair of purely imaginary roots. Meanwhile, the roots of (2.4) with $\tau=0$, i.e., the roots of the following equation

$$\lambda^2+(2-a\varphi'(u^*))\lambda+1+(a-b)\varphi'(u^*)=0$$

are

$$\lambda_{1,2} = \frac{1}{2}(a\varphi'(u^*) - 2\pm((2 - a\varphi'(u^*))^2 - 4(1 - (a-b)\varphi'(u^*)))^{\frac{1}{2}}).$$

By the Lemma in [8], as τ varies, the sum of the mulitiplicites of the roots of Eq. (2.4) in the open right half-plane can change only if a root appears on, or crosses the imaginary axis. Obviously both λ_1 and λ_2 have negative real parts when (H_1) holds. Since τ_0 is the minimal positive value of τ such that Eq. (2.4) has a pair of purely imaginary roots, we can see that all roots of Eq. (2.4) with $\tau \in [0, \tau_0)$ have strictly negative real parts. Moreover, we can easily prove the Lemma 2.4 by Lemma 2.1–2.3.

Applying Lemma 2.4 to system (2.2), we obtain the following result on stability and bifurcation.

Theorem 2.1. If (H_2) is satisfied, then the zero solution of system (2.2) is asymptotically stable for every $\tau \geq 0$; Suppose (H_1) is satisfied, we have

(i) If $\tau \in [0, \tau_0)$, then the zero solution of system (2.2) is asymptotically stable;

(ii) If $\tau > \tau_0$, then the zero solution of system (2.2) is unstable;

(iii) $\tau_n (n = 0, 1, 2, \ldots)$ are Hopf bifurcation values for system (2.2).

3. Direction and Stability of the Hopf Bifurcation

In Section 2, we obtain some conditions which guarantee that system (2.2) undergoes the Hopf bifurcation at some value of τ. In this section, we will study the direction and stability of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard, Kazarinoff and Wan [10].

For convenience, setting $\tau = \tau_0 + \mu, t = s\tau, x(s\tau) = \bar{x}(s), y(s\tau) = \bar{y}(s)$ and still denoting $x(t) = \bar{x}(s), y(t) = \bar{y}(s)$, then system (2.2) can be written as

$$\left\{\begin{aligned} \frac{dx}{dt} &= (\tau_0 + \mu)((a\varphi'(u^*) - 1)x(t) - by(t-1) \\ &\quad + \tfrac{1}{2!}a\varphi''(u^*)x^2(t) + \tfrac{1}{3!}a\varphi'''(u^*)x^3(t) + \cdots), \\ \frac{dy}{dt} &= (\tau_0 + \mu)(-y(t) + \varphi'(u^*)x(t-1) \\ &\quad + \tfrac{1}{2!}\varphi''(u^*)x^2(t-1) + \tfrac{1}{3!}\varphi'''(u^*)x^3(t-1) + \cdots). \end{aligned}\right. \tag{3.1}$$

The linear part of (3.1) is

$$\begin{cases} \frac{dx}{dt} = (\tau_0+\mu)(a\varphi'(u^*)-1)x(t) - b(\tau+\mu)y(t-1), \\ \frac{dy}{dt} = -(\tau_0+\mu)y(t) + (\tau_0+\mu)\varphi'(u^*)x(t-1). \end{cases} \tag{3.2}$$

Denote the nonlinear part of the right side of system (3.1) by

$$f(\mu, u_t) = (\tau_0+\mu)\begin{pmatrix} \frac{1}{2!}a\varphi''(u^*)x_t^2(0) + \frac{1}{3!}a\varphi'''(u^*)x_t^3(0) + \cdots, \\ \\ \frac{1}{2!}\varphi''(u^*)x_t^2(-1) + \frac{1}{3!}\varphi'''(u^*)x_t^3(-1) + \cdots. \end{pmatrix} \tag{3.3}$$

where

$$u_t(\theta) = \begin{pmatrix} x_t(\theta) \\ y_t(\theta) \end{pmatrix} = \begin{pmatrix} x(t+\theta) \\ y(t+\theta) \end{pmatrix}, \theta \in [-1, 0].$$

Choose the phase space as

$$C = C([-1,0], \mathbb{R}^2).$$

For $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$, define the operator

$$\begin{aligned} L_\mu\phi = {} & (\tau_0+\mu)\begin{pmatrix} a\varphi'(u^*)-1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\ & +(\tau_0+\mu)\begin{pmatrix} 0 & -b \\ \varphi'(u^*) & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}. \end{aligned}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation function $\eta(\theta,\mu)$ in $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta). \tag{3.4}$$

In fact, we choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{pmatrix} a\varphi'(u^*)-1 & 0 \\ 0 & -1 \end{pmatrix}\delta(\theta) - (\tau_0+\mu)\begin{pmatrix} 0 & -b \\ \varphi'(u^*) & 0 \end{pmatrix}\delta(\theta+1),$$

where $\delta(\theta)$ is a Dirac function. Then (3.4) is satisfied.

For $\phi \in C^1([-1,0], \mathbb{R}^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\mu,s)\phi(s), & \theta = 0 \end{cases} \tag{3.5}$$

and

$$R\phi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{3.6}$$

Then system (3.1) is equivalent to

$$\dot{u}_t = A(\mu)u_t + Ru_t. \tag{3.7}$$

For $\psi \in C^1([0,1], (\mathbb{R}^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{3.8}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.9}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(\mu_0)$ and A^* are adjoint operators.

From the second section and transformation $t = s\tau$, we easily find that $\mu = 0$ is the Hopf bifurcation value of system (3.1) and $\pm i\tau_0\omega_0$ are the eigenvalues of $A(0)$, and so of A^*. By direct computation, we obtain the following lemma.

Lemma 3.1. $q(\theta) = (1, \frac{a\varphi'(u^*)-1-i\omega_0}{b} e^{i\tau_0\omega_0})^T e^{i\tau_0\omega_0\theta}$ and $q^*(s) = D(\frac{-1+i\omega_0}{b} e^{-i\tau_0\omega_0}, 1)^T e^{i\tau_0\omega_0 s}$ are the eigenvectors of $A(0)$ and A^* associated with $i\tau_0\omega_0$ and $-i\tau_0\omega_0$, respectively and $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \bar{q}(\theta)\rangle = 0$, where

$$D = \frac{b((2\tau_0+1)a\varphi'(u^*) + (1+\tau_0)\omega_0^2 - 2(1+\tau_0)) + 2b\omega_0(a\tau_0\varphi'(u^*) - 2\tau_0 - 1)i}{((2\tau_0+1)a\varphi'(u^*) + (1+\tau_0)\omega_0^2 - 2(1+\tau_0))^2 + 4\omega_0^2(a\tau_0\varphi'(u^*) - 2\tau_0 - 1)^2} e^{-i\tau_0\omega_0}.$$

Using the same notations as in Hassard, Kazarinoff and Wan [10], we first compute the coordinates to describe the center manifold $\mathbf{C}_0$ at $\mu = 0$. Let u_t be the solution of Eq. (3.1) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle, W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{3.10}$$

On the center manifold $\mathbf{C}_0$ we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+W_{30}\frac{z^3}{6}+\cdots, \qquad (3.11)$$

z and $\bar{z}$ are local coordinates for center manifold $\mathbf{C}_o$ in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We consider only real solutions. For solution $u_t \in \mathbf{C}_0$ of (3.1), since $\mu = 0$, then

$$\begin{aligned}\dot{z}(t) &= i\tau_0\omega_0 z+\bar{q}^*(\theta)f(0,w(z,\bar{z},\theta)+2Re\{zq(\theta)\})\\ &\overset{\text{def}}{=} i\tau_0\omega_0 z+\bar{q}^*(0)f_0,\end{aligned} \qquad (3.12)$$

that is

$$\dot{z}(t)=i\tau_0\omega_0 z(t)+g(z,\bar{z}), \qquad (3.13)$$

where

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots. \qquad (3.14)$$

By (3.10), we get

$$\begin{aligned}u_t(\theta) &= W(t,\theta)+2Re\{z(t)q(\theta)\}\\ &= W_{20}(\theta)\tfrac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\tfrac{\bar{z}^2}{2}\\ &\quad+\left(1,\tfrac{a\varphi'(u^*)-1-i\omega_0}{b}e^{i\tau_0\omega_0}\right)^T e^{i\tau_0\omega_0\theta}z\\ &\quad+\left(1,\tfrac{a\varphi'(u^*)-1+i\omega_0}{b}e^{-i\tau_0\omega_0}\right)^T e^{-i\tau_0\omega_0\theta}\bar{z}+\cdots.\end{aligned} \qquad (3.15)$$

From (3.3) and (3.15), we have

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z})=\bar{q}^*(0)f(0,u_t)\\
&= \tau_0\bar{q}^*(0)\begin{pmatrix}\frac{1}{2!}a\varphi''(u^*)\left(W^{(1)}(0)+z+\bar{z}\right)^2+\frac{1}{3!}a\varphi'''(u^*)\left(W^{(1)}(0)+z+\bar{z}\right)^3+\cdots\\ \frac{1}{2!}\varphi''(u^*)\left(W^{(1)}(-1)+ze^{-i\tau_0\omega_0}+\bar{z}e^{i\tau_0\omega_0}\right)^2+\frac{1}{3!}\varphi'''(u^*)\\ \times\left(W^{(1)}(-1)+ze^{-i\tau_0\omega_0}+\bar{z}e^{i\tau_0\omega_0}\right)^3+\cdots\end{pmatrix}\\
&= \tau_0\overline{D}e^{i\tau_0\omega_0}\tfrac{-1-i\omega_0}{b}\Bigg(\tfrac{1}{2!}a\varphi''(u^*)\left(W_{20}^{(1)}(0)\tfrac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\tfrac{\bar{z}^2}{2}+\cdots+z+\bar{z}\right)^2\\
&\quad+\tfrac{1}{3!}a\varphi'''(u^*)\left(W_{20}^{(1)}(0)\tfrac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\tfrac{\bar{z}^2}{2}+\cdots+z+\bar{z}\right)^3+\cdots\Bigg)\\
&\quad+\tau_0\overline{D}\Bigg(\tfrac{1}{2!}\varphi''(u^*)\left(W_{20}^{(1)}(-1)\tfrac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\tfrac{\bar{z}^2}{2}+\cdots+ze^{-i\tau_0\omega_0}+\bar{z}e^{i\tau_0\omega_0}\right)^2\\
&\quad+\tfrac{1}{3!}\varphi'''(u^*)\left(W_{20}^{(1)}(-1)\tfrac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\tfrac{\bar{z}^2}{2}+\cdots+ze^{-i\tau_0\omega_0}+\bar{z}e^{i\tau_0\omega_0}\right)^3+\cdots\Bigg).
\end{aligned} \qquad (3.16)$$

Comparing the coefficients of the both sides of (3.16), we have

$$
\begin{aligned}
g_{20} &= \tau_0\overline{D}\Big(\frac{-1-i\omega_0}{b}a\varphi''(u^*)e^{i\tau_0\omega_0}+\varphi''(u^*)e^{-2\tau_0\omega_0 i}\Big),\\
g_{11} &= \tau_0\overline{D}\Big(\frac{-1-i\omega_0}{b}a\varphi''(u^*)e^{i\tau_0\omega_0}+\varphi''(u^*)\Big),\\
g_{02} &= \tau_0\overline{D}\Big(\frac{-1-i\omega_0}{b}a\varphi''(u^*)e^{i\tau_0\omega_0}+\varphi''(u^*)e^{2\tau_0\omega_0 i}\Big),\\
g_{21} &= \tau_0\overline{D}\Big(\Big(a\varphi''(u^*)\Big(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\Big)+a\varphi'''(u^*)\Big)\frac{-1-i\omega_0}{b}e^{i\tau_0\omega_0}\\
&\quad +\varphi''(u^*)\Big(W_{20}^{(1)}(-1)e^{i\tau_0\omega_0}+2W_{11}^{(1)}(-1)e^{-i\tau_0\omega_0}\Big)\\
&\quad +\varphi'''(u^*)e^{-i\tau_0\omega_0}\Big).
\end{aligned}
\tag{3.17}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we need to compute further. From (3.7) and (3.10) , we have

$$
\begin{aligned}
\dot{W} &= \dot{u}_t-\dot{z}q-\dot{\bar{z}}\bar{q}\\
&= \begin{cases} AW-2Re\{\bar{q}^*(0)f_0q(\theta)\}, & \theta\in[-1,0),\\ AW-2Re\{\bar{q}^*(0)f_0q(0)\}+f_0, & \theta=0.\end{cases}\\
&\overset{\text{def}}{=} AW+H(z,\bar{z},\theta),
\end{aligned}
\tag{3.18}
$$

where

$$
H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}\frac{\bar{z}^2}{2}+\cdots. \tag{3.19}
$$

Substituting the corresponding series into (3.18) and comparing the coefficients, we obtain

$$
(A-2i\tau_0\omega_0)W_{20}(\theta)=-H_{20}(\theta),\ AW_{11}(\theta)=-H_{11}(\theta),\ldots. \tag{3.20}
$$

From (3.18), we know that for $\theta\in[-1,0)$

$$
\begin{aligned}
H(z,\bar{z},\theta) &= -\bar{q}^*(0)f_0q(\theta)-q^*(o)\bar{f}_0\bar{q}(\theta)\\
&= -gq(\theta)-\bar{g}\bar{q}(\theta).
\end{aligned}
\tag{3.21}
$$

Comparing the coefficients with (3.19) gives that

$$
H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta), \tag{3.22}
$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.23}$$

From (3.20) and (3.22), we get

$$\dot{W}_{20}(\theta) = 2i\tau_0\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Notice that $q(\theta) = q(\theta)e^{i\tau_0\omega_0\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}}{3\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta}. \tag{3.24}$$

Similarly, from (3.20) and (3.23), we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta),$$

and

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}}{\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} + E_2. \tag{3.25}$$

It remains to find E_1 and E_2 in (3.24) and (3.25), respectively. From the definition of A and (3.20), we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) \;=\; 2i\tau_0\omega_0 W_{20}(0) - H_{20}(0), \tag{3.26}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) \;=\; -H_{11}(0), \tag{3.27}$$

where $\eta(\theta) = \eta(0, \theta)$.

From (3.18), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_0 \begin{pmatrix} a\varphi''(u^*) \\ \varphi''(u^*)e^{-2i\tau_0\omega_0} \end{pmatrix}, \tag{3.28}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_0 \begin{pmatrix} a\varphi''(u^*) \\ \varphi''(u^*) \end{pmatrix}. \tag{3.29}$$

Substituting (3.24) and (3.28) into (3.26), we have

$$(2i\tau_0\omega_0 I - \int_{-1}^{0} e^{2i\tau_0\omega_0\theta} d\eta(\theta))E_1 = \begin{pmatrix} \tau_0 a\varphi''(u^*) \\ \tau_0\varphi''(u^*)e^{-2i\tau_0\omega_0} \end{pmatrix},$$

that is

$$\begin{pmatrix} 2\tau_0\omega_0 i - \tau_0 a\varphi'(u^*) + \tau_0 & b\tau_0 \\ -\tau_0\varphi'(u^*)e^{-2\tau_0\omega_0 i} & 2\tau_0\omega_0 i + \tau_0 \end{pmatrix} E_1 = \begin{pmatrix} \tau_0 a\varphi''(u^*) \\ \tau_0\varphi''(u^*)e^{-2i\tau_0\omega_0} \end{pmatrix}.$$

Hence, we get

$$E_1^{(1)} = \frac{2a\varphi''(u^*)\omega_0 i + a\varphi''(u^*) - b\varphi''(u^*)e^{-4\tau_0\omega_0 i}}{-4\omega_0^2 + (2 - a\varphi'(u^*))2\omega_0 i + 1 - a\varphi'(u^*) + b\varphi'(u^*)e^{-4\tau_0\omega_0 i}},$$

and

$$E_1^{(2)} = \frac{2\omega_0\varphi''(u^*)ie^{-2\tau_0\omega_0 i} + (1 - a\varphi'(u^*))\varphi''(u^*)e^{-2\tau_0\omega_0 i} - a\varphi'(u^*)\varphi''(u^*)e^{-2\tau_0\omega_0 i}}{-4\omega_0^2 + (2 - a\varphi'(u^*))2\omega_0 i + 1 - a\varphi'(u^*) + b\varphi'(u^*)e^{-4\tau_0\omega_0 i}}.$$

Similarly, substituting (3.25) and (3.29) into (3.27), we get

$$\int_{-1}^{0} d\eta(\theta)E_2 = -\begin{pmatrix} \tau_0 a\varphi''(u^*) \\ \tau_0\varphi''(u^*) \end{pmatrix},$$

that is

$$\begin{pmatrix} \tau_0(a\varphi'(u^*) - 1) & -\tau_0 b \\ \tau_0\varphi'(u^*) & -\tau_0 \end{pmatrix} E_2 = \begin{pmatrix} -\tau_0 a\varphi''(u^*) \\ -\tau_0\varphi''(u^*) \end{pmatrix}.$$

Hence, we have

$$E_2^{(1)} = \frac{(a-b)\varphi''(u^*)}{1 + (b-a)\varphi'(u^*)},$$

and

$$E_2^{(2)} = \frac{\varphi''(u^*)}{1 + (b-a)\varphi'(u^*)}.$$

By the above analysis, we can see that g_{ij} in (3.17) is determined by the parameters and delay in (2.2). Thus, we can compute the following quantities:

$$\begin{aligned} c_1(0) &= \frac{i}{2\tau_0\omega_0}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ \mu &= -\frac{Rec_1(0)}{\eta'(\tau_0)}, \\ \beta_2 &= 2Re(c_1(0)). \end{aligned}$$

From Lemma 2.3, we know $\eta'(\tau_0) > 0$. Therefore , we can get the following theorem. (see Hassard, Kazarinoff and Wan [10]).

Theorem 3.1. For system (3.1), if (H_1) is satisfied, then we have the following properties: $\mu = 0$ is the Hopf bifurcation value of system (3.3); μ_2 determines the directions of the Hopf bifurcation:if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical(subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0(\tau < \tau_0)$;$\beta_2$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable(unstable) if $\beta_2 < 0(\beta_2 > 0)$.

4. Global Hopf Bifurcation

In the second section, we studied the existence of the local Hopf bifurcation of system (1.3). In this section, we shall study the global continuation of the local branch bifurcating from the equilibrium point (u^*, v^*) of system (1.3) with $\tau = \tau_k (k \geq 3)$.

The initial condition of (1.3) is

$$\begin{aligned} u(s) &= \psi_1(s), -\tau \leq s \leq 0; \\ v(s) &= \psi_2(s), -\tau \leq s \leq 0, \end{aligned} \tag{4.1}$$

where $\psi(s) = (\psi_1(s), \psi_2(s))^T \in C([-\tau, 0], \mathbb{R}^2)$.

For convenience, we need some notation. First (1.3) can be written as

$$\dot{x} = F(x_t, \tau), \tag{4.2}$$

where $x = (u, v)^T, x_t(\theta) = x(t+\theta) \in C([-1, 0], \mathbb{R}^2)$. Let $x^* = (u^*, v^*)^T$ be an equilibrium point of (4.2). We set $\mathbf{X} = C([-\tau, 0], \mathbb{R}^2)$, $\sum = Cl\{(x, \tau, p) \in X \times \mathbb{R} \times \mathbb{R}^+ ; x$ is a p-periodic solution of (4.2) which satisfies the initial condition (4.1)$\}$, $N = \{(\bar{x}, \tau, p); F(\bar{x}, \tau, p) = 0\}$.

Let $C_{(x^*, \tau_k, \frac{2\pi}{\omega_0})}$ be the connected component of $(x^*, \tau_k, \frac{2\pi}{\omega_0})$ in $\sum$, where τ_k and ω_0 are defined by (2.5) and (2.6), respectively.

$$\Delta(x^*, \tau, p)(\lambda) = \lambda^2 + (2 - a\varphi'(u^*))\lambda + 1 - a\varphi'(u^*) + b\varphi'(u^*)e^{-2\lambda\tau}. \tag{4.3}$$

Denote the following system

$$\begin{cases} \frac{du}{dt} = & -u(t) + a\varphi(u(t)) - bv(t) + c, \\ \frac{dv}{dt} = & -v(t) + \varphi(u(t)), \end{cases}$$

as $(1.3)_0$. We have

Lemma 4.1 ([7]). When $\tau = 0$, (1.3) can be written as $(1.3)_0$, which is an ordinary differential system. We obtain

(1) For $2 \geq a$, the system of $(1.3)_0$ has no periodic orbits.

(2) Suppose $2 < a$ and $b \leq \frac{a}{2}$. Then for $c \in (\bar{c}_2, \bar{c}_1)$, the system $(1.3)_0$ has no periodic orbits, where $\bar{c}_i$'s satisfy the following condition.

$$\begin{cases} 2 - a\varphi'(\bar{u}_i^*) = 0, \\ \bar{c}_i = \bar{u}_i - (a-b)\varphi(\bar{u}_i). \end{cases} \tag{4.4}$$

From Lemma 4.1, we can obtain

Corollary 4.2. For system (1.3), we have

(1) For $2 \geq a$, the system (1.3) has no nontrivial τ -periodic solution.

(2) If $a > 2$ and $b \leq \frac{a}{2}$, then for $c \in (\bar{c}_2, \bar{c}_1)$, the system (1.3) has no nontrivial τ-periodic solution.

Lemma 4.3. The solutions of system (1.3) with the initial condition (4.1) are ultimately uniformly bounded.

Proof . Suppose $(u(t), v(t))$ be a solution of (1.3) with the initial condition (4.1). Then from the second equation of (1.3) and $\varphi(u_1) = \frac{1}{1+exp(-4u_1)}$, we have

$$\begin{aligned} \mid v(t) \mid &= \mid \exp(-t)(\psi_2(0) + \int_0^t \exp(s)\varphi(u(s-\tau))ds \mid \\ &\leq \exp(-t)(\mid \psi_2(0) \mid + \mid \int_0^t \exp(s)ds \mid) \\ &= (\mid \psi_2(0) \mid -1)\exp(-t) + 1 \\ &\to 1(t \to +\infty). \end{aligned}$$

Hence there exist $T_1 > 0$ and $0 < \epsilon_1 \ll 1$ such that

$$\mid v(t) \mid < 1 + \epsilon_1 = M_1 \text{ for } t > T_1.$$

By the first equation of (1.3), we have

$$\begin{aligned}
|u(t)| &= |\exp(-t)(\psi_1(0)+\int_0^t \exp(s)(a\varphi(s)-bv(s-\tau)+c)ds)| \\
&\leq \exp(-t)(|\psi_1(0)|+\int_0^{T_1+\tau} \exp(s)|a\varphi(s)-bv(s-\tau)+c|ds) \\
&\quad +\exp(-t)\int_{T_1+\tau}^t \exp(s)|a\varphi(s)-bv(s-\tau)+c|ds \\
&\leq \exp(-t)(|\psi_1(0)|+\int_0^{T_1+\tau} \exp(s)|a\varphi(s)-bv(s-\tau)+c|ds) \\
&\quad +\exp(-t)\int_{T_1+\tau}^t \exp(s)(a+bM_1+|c|)ds \\
&= \exp(-t)(|\psi_1(0)|+\int_0^{T_1+\tau} \exp(s)|a\varphi(s)-bv(s-\tau)+c|ds) \\
&\quad +(a+bM_1+|c|)-(a+bM_1+|c|)\exp(T_1+\tau-t) \\
&\to a+bM_1+|c|(t\to+\infty).
\end{aligned}$$

Hence there exist $T > T_1 + \tau > 0$ and $0 < \epsilon_2 \ll 1$ such that

$$| u(t) |< a + bM_1 + \epsilon_2 = M_2 \text{for } t > T.$$

Hence, for the solution $(u(t), v(t))$ of (1.3) with the initial condition (4.1), there is a $T > 0$, such that for $t > T$,

$$| (u(t), v(t)) |\leq| u(t) | + | v(t) |< M_1 + M_2.$$

This implies that the proof of Lemma 4.3 is completed. By Lemma 4.3, we get

Corollary 4.4. For (1.3), if the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto the τ-space is bounded, then the periodic solutions of (1.3) on the $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ are uniformly bounded.

Theorem 4.1. Suppose (H_1) is satisfied. We have

(i) If $0 < a \leq 2$, then there exists a $\bar{\tau}(0 < \bar{\tau} \leq \tau_k, k \geq 3)$ such that for $\tau > \bar{\tau}(k = 3, 4, \ldots)$, system (1.3) has at least a nontrivial periodic solution. In particular, for $\tau > \tau_k$, system (1.3) has at least $k - 2$ nontrivial periodic solutions;

(ii) If $a > 2$ and $b \leq \frac{a}{2}$, then for every $c \in (\bar{c}_2, \bar{c}_1)$, there exists a $\bar{\tau}(0 < \bar{\tau} \leq \tau_k, k = 3, 4, \ldots)$ such that for $\tau > \bar{\tau}$ system (1.3) has at least a nontrivial periodic solution. In particular, for $\tau > \tau_k$, system (1.3) has at least $k-2$ nontrivial periodic solutions, where $\bar{c}_i (i = 1, 2)$ are given by (4.4).

Proof. $(\bar{x}, \bar{\tau}, \bar{p})$ is called a center if $F(\bar{x}, \bar{\tau}) = 0$ and $\Delta(\bar{x}, \bar{\tau}, \bar{p})(\frac{2\pi}{p} i) = 0$. A center $(\bar{x}, \bar{\tau}, \bar{p})$ is said to be isolated if it is the only center in some neighborhood of $(\bar{x}, \bar{\tau}, \bar{p})$. By the proof of Lemma 2.2 and 2.3, there exist $\epsilon > 0, \delta > 0$ and smooth curve $\lambda : (\tau_n - \delta, \tau_n + \delta) \to C$ such that $\Delta(\lambda(\tau)) = 0, \mid \lambda(\tau) - \omega_0 i \mid < \epsilon$ for all $\tau \in [\tau_n - \delta, \tau_n + \delta]$, and

$$\lambda(\tau_n) = \omega_0 i, \frac{dRe\lambda(\tau)}{d\tau} \mid_{\tau = \tau_n} > 0.$$

Let

$$\Omega_{\epsilon, \frac{2\pi}{\omega_0}} = \{(\eta, p); 0 < \eta < \epsilon, \mid p - \frac{2\pi}{\omega_0} \mid < \epsilon\}.$$

Clearly, if $\mid \tau - \tau_n \mid \leq \delta$ and $(\eta, p) \in \partial\Omega_{\epsilon, \frac{2\pi}{\omega_0}}$ such that

$$\Delta(x^*, \tau, p)(\eta + \frac{2\pi}{p} i) = 0 \text{ if and only if } \eta = 0, \tau = \tau_n, p = \frac{2\pi}{\omega_0}.$$

This justifies (A_4) for $m = 1$ in [3]. From the above discussion, it is obvious that the hypotheses of $(A_1) - (A_4)$ in J. Wu[3] are satisfied. Moreover, if we put

$$H^{\pm}(x^*, \tau_n, \frac{2\pi}{\omega_0})(\eta, p) = \Delta(x^*, \tau_n \pm \delta, p)(\eta + i\frac{2\pi}{p}),$$

then we have the cross number

$$\begin{aligned} \gamma\Big(x^*, \tau_n, \frac{2\pi}{\omega_0}\Big) &= \deg_B\bigg(H^-\Big(x^*, \tau_n, \frac{2\pi}{\omega_0}\Big), \Omega_{\epsilon, \frac{2\pi}{\omega_0}}\bigg) \\ &\quad -\deg_B\bigg(H^+\Big(x^*, \tau_n, \frac{2\pi}{\omega_0}\Big), \Omega_{\epsilon, \frac{2\pi}{\omega_0}}\bigg) \\ &= -1. \end{aligned}$$

By Theorem 3.3 of J. Wu[3], we conclude that the connected component $C_{(x^*, \tau_k, \frac{2\pi}{\omega_0})}$ through $(x^*, \tau_n, \frac{2\pi}{\omega_0})$ in $\sum$ is nonempty. Meanwhile, we have

$$\sum_{(\bar{x}, \bar{\tau}, \bar{p}) \in C_{(x^*, \tau_n, \frac{2\pi}{\omega_0})}} \gamma(\bar{x}, \bar{\tau}, \bar{p}) < 0,$$

and hence $C_{(x^*,\tau_n,\frac{2\pi}{\omega_0})}$ is unbounded. From

$$\tau_k = \frac{1}{2\omega_0}\sin^{-1}\left(\frac{(2-a\varphi'(u^*))\omega_0}{b\varphi'(u^*)} + 2k\pi\right) > \frac{2\pi}{\omega_0}, \text{ for } k \geq 3$$

we have

$$\frac{1}{n_k} < \frac{2\pi}{\tau_k\omega_0} < 1$$

for some $n_k \in N$. From the continuous change of p in the connected component and Corollary 4.2, we can show that, if $(x,\tau,p) \in C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$, then $\frac{\tau}{n_k} < p < \tau$. This implies that if the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto τ -space is bounded, then the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto p-space is bounded. By Corollary 4.4, the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto x-space is also bounded. This shows that if $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ is to be unbounded, the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto τ-space must be undounded. Also, note that Lemma 4.1 implies that the system (1.3) with $\tau = 0$ has non-constant periodic solution under the condition of Theorem 4.5. Consequently, the projection of $C_{(x^*,\tau_k,\frac{2\pi}{\omega_0})}$ onto τ-space must be an interval $(\bar{\tau}, \infty)$ with $0 < \bar{\tau} \leq \tau_k$. Finally, notice that the connected components of every isolated center in $\sum$ don't intersect one another. This shows that for each $\tau > \tau_k (k = 3, 4, \ldots)$ (1.3) has at least $k-2$ non-constant periodic solutions with periods in $(\frac{\tau}{n_k}, \tau)$. This completes the proof.

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Stabilization Role of Self-Inhibitory Cohen–Grossberg Neural Networks with General Activation Functions*

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Abstract

Without assuming differentiability and monotonicity of the activation functions and any symmetry of interconnections, we establish some sufficient conditions for the global asymptotic stability of a unique equilibrium for Cohen-Grossberg neural network. The results show that self-inhibitory connections play a role of stabilizing a network.

1. Introduction

The Hopfield neural network is an artificial feedback neural network described by a system of ordinary differential equations

$$C_i\dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^{n} t_{ij}s_j(x_j) + I_i, \qquad i = 1, 2, \ldots, n. \tag{1.1}$$

Here $n \geq 2$ is the number of neurons in the network, x_i denotes the state variable associated with the i-th neuron, the positive constants C_i's and R_i's are the neuron amplifier input capacitances and resistences, respectively, and

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I_i's are the constant inputs from outside of the network. The $n \times n$ connection matrix $T = (t_{ij})$ tells how the neurons are connected in the network, and the activation functions s_j's show how neurons respond to each other. As a generalization, the Cohen-Grossberg neural network was proposed by Cohen and Grossberg [3] in 1983, which is of the form

$$\dot{x}_i = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{n} t_{ij} s_j(x_j)\right], \qquad i = 1, \ldots, n, \tag{1.2}$$

where a_i represents an amplification function, and b_i is an appropriately behaved function. s_j's and $T = (t_{ij})$ are the same as in (1.1). Functions a_i's, b_i's and s_i's are subject to certain conditions to be specified later.

Due to their promising potential for the tasks of classification, associative memory, parallel computations, and their ability to solve difficult optimization problems, (1.1) and (1.2) have attracted great attention of the scientific community. Various generalizations and modifications of (1.1) and (1.2) have also been proposed and studied. See, for example, Bélair [1], Cao and Wu [2], Gopalsamy and He [4], van den Driessche and Zou [11], Lu [7], Matsuoka [9], Guan, Chen, and Qin [5] and Ye, Nichel, and Wang [13].

In applications of neural networks to parallel computation, signal processing and other problems involving the solutions of optimization problems, it is required that there be a well-defined computable solution for all possible initial states. In other words, it is required that the network should have a unique equilibrium that is globally attractive. In fact, earlier applications of neural networks to optimization problems have suffered from the existence of a complicated set of equilibria (see Tank and Hopfield [10]). Thus, the global attractivity of a unique equilibrium for the system is of great importance for both practical and theoretical purposes, and has been the major concern of many authors. For the Hopfield type neural networks, Bélair [1], Cao and Wu [2], Gopalsamy and He [4], van den Driessche and Zou [11], Lu [7], Matsuoka [9] Guan,Chen, and Qin [5] obtained some nice results. However, for the Cohen–Grossberg neural networks, little has been done in the literature. Thus, the purpose of this paper is to generalize some known results in the Hopfield networks to the Cohen–Grossberg neural networks with general activation functions, that is, the following system

$$\dot{x}_i(t) = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t)) + J_i\right], \qquad i = 1, 2, \ldots, n, \tag{1.3}$$

where J_i's denote the constant inputs from outside of the system. In fact, we focus on the cases where neurons have inhibitory self-connections, and we do not require any symmetry for the connections and thus allow much broader connection topologies for the network. Note that, the form of (1.3) also includes many other model systems from population biology.

This paper is organized as follows. In Section 2, we introduce some necessary notations and assumptions. In Section 3, we establish our main results on the stability of (1.3). Some examples are also given in Section 3 to demonstrate the main results.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}$. If $x \in \mathbb{R}^n$, then $x^T = (x_1, \ldots, x_n)$ denotes the transpose of x, $\|x\|_2 = (x^T x)^{1/2}$. Let $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ real matrices. For $Z \in \mathbb{R}^{n\times n}$, the spectral norm of Z is defined as $\|Z\|_2 = \left(\max\{|\lambda| : \lambda \text{ is an eigenvalue of } Z^T Z\}\right)^{1/2}$.

The initial conditions associated with (1.3) are of the form

$$x_i(s) = x_i(0), \qquad i = 1, 2, \ldots, n. \tag{2.1}$$

We list some assumptions which will be used in our main results.

(H_1) For each $i \in \{1, 2, \ldots, n\}$, a_i is positive and continuous.

(H_2) For each $i \in \{1, 2, \ldots, n\}$, a_i is bounded, positive and continuous, furthermore we assume $0 < \underline{\alpha}_i \leq a_i(u) \leq \overline{\alpha_i}$.

(H_3) For each $i \in \{1, 2, \ldots, n\}$, $b_i \in C^1$ and $b_i'(u) > 0$.

(H_4) For each $i \in \{1, 2, \ldots, n\}$, $b_i \in C^1$ and $\inf b_i'(u) \geq m_i > 0$.

For the activation functions $s_i(x)$, $i = 1, 2, \ldots, n$, they are typically assumed to be sigmoid which implies that they are monotone and smooth (see, e.g., Assumption A in [13]), that is, they are required to satisfy the following

(A_1) $s_i \in C^1(\mathbb{R})$, $s_i'(x) > 0$ for $x \in \mathbb{R}$ and $s_i(0) = \sup_{x\in\mathbb{R}} s_i'(x) > 0$, $i = 1, 2, \ldots, n$.

(A_2) $s_i(0) = 0$ and $\lim_{x\to\pm\infty} s_i(x) = \pm 1$.

But as pointed out in [11], for some purposes of networks, nonmonotonic and not necessarily smooth functions might be better candidates for neuron activation in designing and implementing an artificial neural network. Note that many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input-output functions are frequently adopted. This suggests a modification of (A_1) and (A_2) to the following

(H_5) For each $i \in \{1,2,\dots,n\}$, $s_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with Lipschitz constant L_i, i.e., $|s_i(u_i)-s_i(v_i)| \le L_i|u_i-v_i|$ for all $u_i, v_i \in \mathbb{R}$;

(H_6) For each $i \in \{1,2,\dots,n\}$, $|s_i(x)| \le M_i$, $x \in \mathbb{R}$ for some constant $M_i > 0$.

Note that unlike in [3], T is not required to be symmetric in this paper, which means that our results will be applicable to networks with much broader connection structures.

3. Main Results

3.1. Existence of Equilibrium

Theorem 3.1. If (H_1),(H_3) and (H_6) hold, then for every input J, there exists an equilibrium for system (1.3).

Proof. For every fixed input J, by (H_1), we know that x^* is an equilibrium of (1.3) if and only if $x^* = (x_1^*, \cdots, x_n^*)^T$ is a solution of equations

$$b_i(x_i) - \sum_{j=1}^{n} t_{ij}s_j(x_j) + J_i = 0, i = 1, \dots, n. \tag{3.1}$$

We just need to show that (3.1) has a solution. For every fixed input J,

$$\left|\sum_{j=1}^{n} t_{ij}s_j(u_j) + J_i\right| \le \sum_{j=1}^{n} |t_{ij}|M_j + |J_i| := P_i,$$

by (H_3), we know b_i^{-1} is also increasing. Now consider

$$x_i^* = h_i(x_1^*, x_2^*, \dots, x_n^*) = b_i^{-1}\left(\sum_{j=1}^{n} t_{ij}s_j(u_j) + J_i\right) \text{ for } i = 1, 2, \dots, n.$$

Then we have

$$|h_i(x_1^*, x_2^*, \dots, x_n^*)| \le b_i^{-1}(P_i) := D_i, \text{ for } i = 1, 2, \dots, n.$$

It follows that $(h_1, h_2, \dots, h_n)^T$ maps a bound set $D = D_1 \times D_2 \times \cdots \times D_n$ to itself. Then by Brouwer's fixed point theorem, we get the existence of the equilibrium. The proof is complete.

Theorem 3.2. If (H_1),(H_4), (H_5) and (H_6) hold, in addition, if

$$m_i > L_i \sum_{j=1}^{n} |t_{ji}| \tag{3.2}$$

then there is a unique equilibrium for system (1.3).

Proof. The existence of the equilibrium follows from the above theorem. So we just need to prove the uniqueness. Assume that $x^* = (x_1, x_2, \dots, x_n)^T$ and $y^* = (y_1, y_2, \dots, y_n)^T$ are two equilibria of (1.3), then we have

$$\begin{aligned} m_i|x_i^* - y_i^*| &\le |b_i(x_i^*) - b_i(y_i^*)| \\ &= \Big|\sum_{j=1}^{n} t_{ij}(s_j(x_j^*) - s_j(y_j^*))\Big| \\ &\le \sum_{j=1}^{n} |t_{ij}| L_j |x_j^* - y_j^*|, \end{aligned}$$

and hence

$$\begin{aligned} \sum_{i=1}^{n} m_i|x_i^* - y_i^*| &\le \sum_{i=1}^{n}\sum_{j=1}^{n} |t_{ij}| L_j |x_j^* - y_j^*| \\ &= \sum_{i=1}^{n} L_i \sum_{j=1}^{n} |t_{ji}||x_i^* - y_i^*|, \end{aligned}$$

which leads to

$$\sum_{i=1}^{n}\Big(m_i - L_i \sum_{j=1}^{n} |t_{ji}|\Big)|x_i^* - y_i^*| \le 0.$$

But this is impossible if $x^* \neq y^*$. Thus the proof is complete.

3.2. Global Asymptotic Stability

Let x^* be an equilibrium of (1.3) and $u(t) = x(t) - x^*$. Substituting $x(t) = u(t) + x^*$ into (1.3) leads to

$$\dot{u}_i(t) = -a_i(u_i + x_i^*)\left[b_i(u_i + x_i^*) - b_i(x_i^*) - \sum_{j=1}^{n} t_{ij}\left(s_j(u_j + x_j^*) - s_j(x_j^*)\right)\right]. \tag{3.3}$$

By the definition of x^*, (3.3) further reduces to

$$\dot{u}_i(t) = -\alpha_i(u_i(t))\left[\beta_i(u_i(t)) - \sum_{j=1}^{n} t_{ij} g_j(u_j)\right], \quad i = 1, 2, \ldots, n, \tag{3.4}$$

where $\alpha_i(u_i(t)) = a_i(u_i(t) + x_i^*)$, $\beta_i(u_i(t)) = b_i(u_i(t) + x_i^*) - b_i(x_i^*)$, $g_j(u_j(t)) = s_j(u_j(t) + x_j^*) - s_j(x_j^*)$. If we let $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, $A(u) = \operatorname{diag}\{\alpha_1(u_1), \ldots, \alpha_n(u_n)\}$, $B(u) = (\beta_1(u_1), \ldots, \beta_n(u_n))^T \in \mathbb{R}^n$, $T = [t_{ij}]_{n\times n}$, $g(u) = (g_1(u_1), \ldots, g_n(u_n))^T$, then system (3.4) can be rewritten as

$$\dot{u} = -A(u)\left(B(u) - Tg(u(t))\right). \tag{3.5}$$

It is obvious that x^* is globally asymptotically stable for (1.3) if and only if the trivial solution $u = 0$ of (3.4) or (3.5) is globally asymptotically stable. Moreover, the uniqueness of the equilibrium of (1.3) follows from its global asymptotic stability.

Theorem 3.3. Suppose (H_1), (H_3), (H_5) and (H_6) are satisfied. Assume also that for each $i \in \{1, 2, \ldots, n\}$,

$$0 < \gamma_i \le \frac{\beta_i(u)}{u}, \text{ for } u \neq 0 \text{ and some } \gamma_i > 0, \tag{3.6}$$

and

$$ug_i(u) > 0 \text{ when } u \neq 0. \tag{3.7}$$

Then, for every input J, if one of the following conditions

$$\lambda_{\max}\left(\frac{T + T^T}{2}\right) < \frac{\gamma_i}{L_i}, \qquad \text{for each } i \in \{1, 2, \ldots, n\}, \tag{3.8}$$

$$t_{ii} + \frac{1}{2}\sum_{j\neq i} |t_{ij} + t_{ji}| < \frac{\gamma_i}{L_i}, \qquad \text{for each } i \in \{1, 2, \ldots, n\}, \tag{3.9}$$

holds, system (1.3) has a unique equilibrium x^*, which is globally asymptotically stable.

Proof. The existence of equilibrium follows from Theorem 3.1. From (3.7) and (H_5), we can get

$$0 < g_i(u_i)u_i = (s_i(u_i + x_i^*) - s_i(x_i^*))u_i \leq L_i u_i^2,$$

and so we have

$$g_i^2(u_i) \leq L_i u_i g_i(u_i). \tag{3.10}$$

Now, let $V(u) = \sum_{i=1}^n \int_0^{u_i} \frac{g_i(s)}{\alpha_i(s)} ds$. Note that if $u_i = 0$, then $g_i(u_i)\beta_i(u_i) = 0 = \frac{\gamma_i}{L_i} g_i^2(u_i)$; and if $u_i \neq 0$, then $g_i(u_i)\beta_i(u_i) = g_i(u_i)u_i \frac{\beta_i(u_i)}{u_i} \geq \frac{\gamma_i}{L_i} g_i^2(u_i)$. Hence, we have $g_i(u_i)\beta_i(u_i) \geq \frac{\gamma_i}{L_i} g_i^2(u_i)$ for all $u_i \in \mathbb{R}$. Therefore

$$\begin{aligned}
D_{(3.5)}V(u) &= \sum_{i=1}^n \frac{g_i(u_i)}{\alpha_i(u_i)} \frac{du_i}{dt} \\
&= \sum_{i=1}^n -g_i(u_i)\Big[\beta_i(u_i) - \sum_{j=1}^n t_{ij} g_j(u_j)\Big] \\
&= \sum_{i=1}^n -g_i(u_i)u_i \frac{\beta_i(u_i)}{u_i} + \sum_{i=1}^n g_i(u_i) \sum_{j=1}^n t_{ij} g_j(u_j) \\
&\leq \sum_{i=1}^n -\frac{\gamma_i}{L_i} g_i^2(u_i) + \sum_{i=1}^n g_i(u_i) \sum_{j=1}^n t_{ij} g_j(u_j).
\end{aligned}$$

So if (3.8) holds, then we have

$$\begin{aligned}
D_{(3.5)}V(u) &\leq \sum_{i=1}^n -\frac{\gamma_i}{L_i} g_i^2(u_i) + \lambda_{\max}\Big(\frac{T+T^T}{2}\Big) \sum_{j=1}^n g_i^2(u_j) \\
&= \sum_{i=1}^n \Big(-\frac{\gamma_i}{L_i} + \lambda_{\max}\Big(\frac{T+T^T}{2}\Big)\Big) g_i^2(u_i) \\
&= -\sum_{i=1}^n \Big(\frac{\gamma_i}{L_i} - \lambda_{\max}\Big(\frac{T+T^T}{2}\Big)\Big) g_i^2(u_i) \\
&\leq 0.
\end{aligned}$$

If (3.9) holds, then we have

$$\begin{aligned}
D_{(3.5)}V(u) &\leq \sum_{i=1}^{n} -\frac{\gamma_i}{L_i} g_i^2(u_i) + \frac{1}{2}\sum_{i,j}^{n} g_i(u_i)(t_{ij}+t_{ji})g_j(u_j) \\
&\leq \sum_{i=1}^{n} -\frac{\gamma_i}{L_i} g_i^2(u_i) + \sum_{i=1}^{n} t_{ii} g_i^2(u_i) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} |t_{ij}+t_{ji}||g_i(u_i)||g_j(u_j)| \\
&\leq \sum_{i=1}^{n} -\frac{\gamma_i}{L_i} g_i^2(u_i) + \sum_{i=1}^{n} t_{ii} g_i^2(u_i) \\
&\quad + \frac{1}{4}\sum_{i,j,i\neq j}^{n} |t_{ij}+t_{ji}|\Big(g_i^2(u_i)+g_j^2(u_j)\Big) \\
&= \sum_{i=1}^{n} -\frac{\gamma_i}{L_i} g_i^2(u_i) + \sum_{i=1}^{n}\left(t_{ii} + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} |t_{ij}+t_{ji}|\right) g_i^2(u_i) \\
&= -\sum_{i=1}^{n}\left[\frac{\gamma_i}{L_i} - \Big(t_{ii} + \frac{1}{2}\sum_{j\neq i}^{n} |t_{ij}+t_{ji}|\Big)\right] g_i^2(u_i) \\
&\leq 0.
\end{aligned}$$

Note that $V(u) \geq 0$ and $V(u) = 0$ if and only if $u = 0$ and $E = \{u \in \mathbb{R}^n | DV(u) = 0\} = \{0\}$. So, by the LaSalle Invariant Principle, the zero solution of (3.5) is globally asymptotically stable, and hence, x^* is globally asymptotically stable for (1.3). The proof is complete.

Remark 3.1. When it comes to Hopfield neural network, unlike as in [5] and [9], in the above theorem, we remove the differentialbility of activation functions s_j and obtain results similar to those of Theorem and Corollary 1 of [9].

Theorem 3.4. Suppose $(H_1), (H_3), (H_5), (H_6)$, (3.6) and (3.7) hold. If

$$t_{ii} + \sum_{j\neq i} |t_{ji}| < \frac{\gamma_i}{L_i}, \text{ for each } i \in \{1, 2, \ldots, n\}. \qquad (3.11)$$

Then, for every input J, system (1.3) has a unique equilibrium which is globally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 3.3 by choosing a different Lyapunov function

$$V(u(t)) = \sum_{i=1}^{n} \int_0^{u_i} \frac{sgn(s)}{\alpha_i(s)} ds. \tag{3.12}$$

Remark 3.2. In the above two theorems, (3.9) and (3.11) show that the self-inhibitory connections do play an important role in stabilizing a network.

Theorem 3.5. Assume that (H_2),(H_3),(H_5),(H_6),(3.6) and (3.7) hold. If

$$\delta \overset{def}{=} L\|T\|_2\eta < 1, \tag{3.13}$$

where $L = \max_{1\le i\le n} L_i$ and $\eta = \max_{1\le i\le n}\{\overline{\alpha}_i\}/\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}$. Then, for any input J, system (3.5) or system (1.3) has a unique equilibrium x^*, which is globally asymptotically stable.

Proof. (H_2), (H_3) and (H_6) imply the existence of the equilibrium x^*. Let $V(u) = \frac{1}{2}\|u\|^2$. Then

$$\begin{aligned}
\frac{dV(u(t))}{dt} &= -u^T A(u)B(u) + u^T A(u)Tg(u) \\
&\le -\sum_{i=1}^{n} \alpha_i(u_i)\beta_i(u_i)u_i + \|u\|_2\|A(u)\|_2\|T\|_2\|g(u)\|_2 \\
&\le -\sum_{i=1}^{n} \underline{\alpha}_i\gamma_i u_i^2 + \|u\|_2 \max_{1\le i\le n}\{\overline{\alpha}_i\}\|T\|_2\|g(u)\|_2 \\
&\le -\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}\sum_{i=1}^{n} u_i^2 + \max_{1\le i\le n}\{\overline{\alpha}_i\}\|T\|_2 L\|u\|_2^2,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{dV(u(t))}{dt} &\le -\left(\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\} - \max_{1\le i\le n}\{\overline{\alpha}_i\}L\|T\|_2\right)\|u\|_2^2 \\
&= -(1-\delta)\left(\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}\right)\sum_{i=1}^{n} u_i^2(t) < 0,
\end{aligned}$$

which follows that the trivial solution of system (3.5) is globally asymptotically stable, and therefore, x^* is globally asymptotically stable for system (1.3). This completes the proof.

Corollary 3.1. If $a(u) \equiv 1$ and $b_i(u) = \gamma_i u$, then the conditions of Theorem 3.5 reduce to

$$L\|T\|_2 < \min_{1\le i\le n} \{\gamma_i\}, \tag{3.14}$$

and (H_5),(H_6). That is, if (H_5),(H_6) and (3.14) hold, then the conclusion of Theorem 3.5 holds.

Theorem 3.6. If all conditions of Theorem 3.2 are satisfied, then the unique equilibrium x^* of (1.3) is globally asymptoticly stable.

Proof. This theorem can be easily proved by using the Lyapunov function (3.12).

3.3. Examples

Example 3.1. Consider the all excitatory doubly stochastic connection matrix studied in Marcus and Westervelt [8] and Wu [12], i.e., $a_i(u) = 1, b_i(u) = u, t_{ii} = 0, t_{ij} = \frac{1}{n-1}$ for $i \neq j, s_j(u) = s(u)$ is increasingly sigmoid with neuron gain $s'(0) = \sup_{x\in\mathbb{R}} s'(x) > 0$. Then, $\|T\| = 1, L = s'(0)$, and all (3.8), (3.9),(3.13),(3.14) and (3.11) reduce to

$$s'(0) < 1. \tag{3.15}$$

Note that (3.15) has been proved by Wu [12] to be a sufficient and necessary condition for such a network to have a unique equilibrium which is the global attractor.

Example 3.2. Consider

$$\frac{dx(t)}{dt} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \sin(\frac{2}{\sqrt{3}}x_1(t)) \\ \sin(\frac{2}{\sqrt{3}}x_2(t)) \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \tag{3.16}$$

Here, $a_i(x_i(t)) = 1$, $b_i(x_i(t)) = x_i(t)$, $s_i(x_i) = \sin(2x_i/\sqrt{3})$ for $i = 1, 2$, $T = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$. All conditions of Theorem 3.5 are satisfied, therefore, system (3.16) has an equilibrium, which is globally asymptotically stable. But the Corollary 2 of [7] can not be applied to system (3.16).

Example 3.3. Consider

$$\begin{aligned}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} &= -\begin{pmatrix}2+\sin x_1 & 0\\ 0 & 2+\cos x_2\end{pmatrix}\\ &\quad\times\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}-\begin{pmatrix}-\frac{\sqrt{2}}{4} & \frac{-\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}\end{pmatrix}\begin{pmatrix}s_1(x_1(t))\\ s_2(x_2(t))\end{pmatrix}\right], \qquad (3.17)\end{aligned}$$

where s_1 and s_2 satisfy (H_6) and (H_7) with $L = 1$. Then $\delta = \frac{3}{4} < 1$, it follows from Theorem 3.5 that (3.17) has a unique equilibrium,which is globally asymptotically stable.

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Asymptotic Behavior of Hopfield Neural Networks with Delays*

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Abstract

This paper deals with asymptotic behavior of Hopfield neural network with delays. Sufficient conditions for determining the attracting sets and basin of attraction of the network are obtained. Examples are given to illustrate the obtained results.

1. Introduction

The study of asymptotic behavior of neural networks has been a very active area in engineering and applied mathematics. Various interesting results on the existence, uniqueness and global asymptotic stability of an equilibrium point have been reported (see, for instance, [1-15]). However, not much has been developed in the direction of giving criteria on the existence of invariant sets and attracting sets for neural networks with delays although significant advance in the existence of invariant sets and attracting sets, and estimates on the basin of attraction has been made for the nonlinear dynamical systems described by ordinary and partial differential equations [16-25]. Hence techniques and methods for determining the invariant sets and attracting set should be developed and explored for neural networks. This paper presents one such

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method which is applied to Hopfield neural networks with delays. The method yields conditions for determining the invariant sets, attracting sets and basin of attraction.

2. Notation and System Description

Let $\mathbb{R}_+^n = \{x | x \in \mathbb{R}^n, x \geq 0\}$, $\Omega \subset \mathbb{R}^n$ be an open region and $\Omega^+ = \{x | x \in \Omega \cap \mathbb{R}_+^n\}$. $C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, $C \triangleq C[[-r, 0], \mathbb{R}^n]$.

Consider the Hopfield neural network system with delays (described in [2]) as follows:

$$\begin{cases} \dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^n a_{ij} g_j(u_j(t - r_{ij}(t))) + p_i(t), & t \geq t_0, \\ u_i(t) = \phi_i(t), \quad t_0 - r \leq t \leq t_0, & i = 1, \ldots, n, \end{cases} \tag{1}$$

where $0 \leq r_{ij}(t) \leq r$, $b_i > 0$ and $a_{ij} \in \mathbb{R}$ are constants.

For convenience, in the following we shall rewrite Eq.(1) in the form:

$$\begin{cases} \dot{u}(t) = -Bu(t) + Ag(u_t) + P(t), \quad t \geq T_0, \\ u(t) = \phi(t), \quad t_0 - r \leq t \leq t_0, \end{cases} \tag{2}$$

in which $u(t) = \text{col}\{u_i(t)\}$, $u_t = u(t + s)$, $-r \leq s \leq 0$, $B = \text{diag}\{b_i\}$, $A = (a_{ij})_{n \times n}$, $r(t) = (r_{ij}(t))$ and $g(u) = \text{col}\{g_j(u_j)\}$ with $g(0) = 0$ are continuous; $P(t) = \text{col}\{p_i(t)\}$ and $\phi = \text{col}\{\phi_i\}$ are also continuous.

For any $\phi \in C$, a solution of (1) is a function u: $\mathbb{R}_+ \to \mathbb{R}^n$ satisfying (1) for $t \geq t_0$. Throughout the paper, we always assume that system (1) has a continuous solution denoted by $u(t, t_0, \phi)$ or simply $u(t)$ if no confusion should occur. A point $u^* \in \mathbb{R}^n$ is called an equilibrium point of (1), if $u(t) = u^*$ is a solution of (1). $A \geq B$ ($A < B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\geq$" ("$<$"). Especially, A is called a nonnegative matrix if $A \geq 0$.

For $u \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, we define $[u(t)]^+ = \text{col}\{|u_i(t)|\}$ and $[A]^+ = (|a_{ij}|)_{n \times n}$. For $u_t \in C$, we define $[u(t)]_r^+ = \text{col}\{\|u_i(t)\|_r\}$, where $\|u_i(t)\|_r = \sup_{-r \leq s \leq 0} |u_i(t + s)|$.

Definition 1. The set $S \subset C$ is called a positive invariant set of (1) if any initial value $\phi \in S$, we have $u(t, t_0, \phi) \in S, t \geq t_0$.

Definition 2. A set $S_1 \subset C$ is called a attracting set of (1), if S_1 possesses an open neighborhood D such that for any initial value $\phi \in D$, the solution $u(t, t_0, \phi)$ converges to S_1 as $t \to +\infty$, that is,

$$\operatorname{dist}(u(t, t_0, \varphi), S_1) \to 0, \quad \text{as} \quad t \to +\infty. \tag{3}$$

The above distance is understood to be the distance of a point to a set

$$\operatorname{dist}(y, S) = \inf_{x \in S} d(y, x),$$

$d(y, x)$ denoting the distance of x to y in C. The open set D is called an attraction basin of S_1. The set S_1 is called a global attracting set of (1) if $D = C$.

Especially, when $S_1 = \{0\}$, the equilibrium point $u(t) = 0$ is said to be asymptotically stable if it attracts solutions $u(t, t_0, \phi)$ for all $\phi \in D$. In this case, D is called the domain of attraction.

3. Main Results

Theorem 1. Assume that there is an axisymmetric and convex region $\Omega \subset \mathbb{R}^n$ such that

$$\begin{aligned} |g_j(u_j(t - r_{ij}(t))| &\leq h_j(\|u_j(t)\|_r), \forall u \in \Omega, \forall t \geq t_0, \\ |p_i(t)| &\leq p_i, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{4}$$

where $h_j(s)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and monotonically nondecreasing in $\mathbb{R}_+$. Let $P = (p_1, \ldots, p_n)^T$, $H([u(t)]_r^+) = [h_1(\|u_1(t)\|_r), \ldots, h_n(\|u_n(t)\|_r)]^T$. Suppose further that there is a region $\Omega^* \subset \Omega$ such that

$$\Delta([x]^+) \triangleq B^{-1}([A]^+ H([x]^+) + P) - [x]^+ < 0, \ \forall x \in \Omega^*. \tag{5}$$

Then, for any $x^* \in \Omega^*$, the set $D = \{\phi \in C_\Omega | [\phi]_r^+ \leq [x^*]^+\}$ is a positive invariant set of (1).

Proof. Let $E = [1, \ldots, 1]^T$, for any $x^* \in \Omega^*$ we will prove that, when $[\phi]_r^+ \leq [x^*]^+$,

$$[u(t)]^+ \leq [x^*]^+, \qquad \text{for } t \geq t_0. \tag{6}$$

By (5), the axisymmetry and convexity of Ω, for the above x^*, there is an $\varepsilon > 0$ such that $[x^*]^+ + E\varepsilon \in \Omega^* \cap \Omega^+$.

To prove (6), we first prove, when $[\phi]_r^+ < [x^*]^+ + \varepsilon E$, that

$$[u(t)]^+ < [x^*]^+ + \varepsilon E \triangleq N \in \Omega^* \cap \Omega^+, \quad \text{for } t \geq t_0. \tag{7}$$

If (7) does not hold, there must be some i, and $t_1 > t_0$, such that

$$|u_i(t_1)| = N_i, \quad |u_i(t)| < N_i, \quad \text{for } t < t_1, \tag{8}$$

and

$$[u(t)]^+ \leq N, \quad \text{for } t_0 \leq t \leq t_1, \tag{9}$$

where N_i is the i-th component of vector N.

On the other hand, by variation of constant formula, system (2) can be written as

$$u(t) = e^{-B(t-t_0)}u(t_0) + \int_{t_0}^{t} e^{-B(t-s)}[Ag(u_t) + P(s)]\, ds. \tag{10}$$

Then, by (4), (8), (9) and the monotonicity of H in Ω^+, we have

$$\begin{aligned}
[u(t_1)]^+ &\leq e^{-B(t_1-t_0)}[u(t_0)]^+ + \int_{t_0}^{t_1} e^{-B(t_1-s)}([A]^+H([u(s)]_r^+) + P(s))\, ds \\
&\leq e^{-B(t_1-t_0)}[\phi]_r^+ + \int_{t_0}^{t_1} e^{-B(t_1-s)}BB^{-1}([A]^+H(N) + P)\, ds \\
&< e^{-B(t_1-t_0)}N + (I - e^{-B(t_1-t_0)})B^{-1}([A]^+H(N) + P). \\
&= e^{-B(t_1-t_0)}(N - B^{-1}([A]^+H(N) + P)) \\
&\quad + B^{-1}([A]^+H(N) + P).
\end{aligned} \tag{11}$$

Since $N \in \Omega^*$, so $\Delta(N) < 0$, and (11) implies that

$$[u(t_1)]^+ < N,$$

that is,

$$|u_i(t_1)| < N_i,$$

which contradicts the first equality in (8), and so (7) holds. Letting $\varepsilon \to 0$, then (6) follows, and we complete the proof.

Theorem 2. Let $\mathcal{N} = \{1, 2, \ldots, n\}$, Δ_i be the ith component of Δ, $\Omega^* = \{x \in \Omega^+ | \Delta(x) < 0\}$, and

$$\Omega_1 = \bigcup_{x^* \in \Omega^*} \{x \in \Omega^+ | x \leq x^*\}, \quad \Omega_2 = \bigcup_{i \in \mathcal{N}} \{x \in \Omega^+ | \Delta_i(x) < 0\}. \qquad (12)$$

If, in addition to Condition (4) in Theorem 1, $r = r(t) \geq 0$ and $\lim_{t\to\infty} (t - r(t)) = \infty$, then the set $D_1 = \{\phi \in C_\Omega | [\phi]_r^+ \in \Omega_1 - \Omega_2\}$ is an attracting set of (1) and the set $D_2 = \{\phi \in C_\Omega | [\phi]_r^+ \in \Omega_1\}$ is a basin of attraction of D_1, where $\Omega_1 - \Omega_2$ is the complement of Ω_2 in Ω_1.

Proof. From Theorem 1, for any $\phi \in D_2$, there always is $x^* \in \Omega^*$ such that

$$[u(t)]^+ \leq x^*, \quad \forall t \geq t_0. \qquad (13)$$

We now prove that

$$\overline{\lim}_{t\to\infty} [u(t)]^+ \in \Omega_1 - \Omega_2. \qquad (14)$$

If it is not true, from (13), there is a constant vector $\sigma \in \Omega_2 \cap \Omega_1$ such that

$$\overline{\lim}_{m\to\infty} [u(t)]^+ = \sigma. \qquad (15)$$

By the definition of superior limit and (15), there exist a sufficiently small constant ε and $t_3 > t_0$ such that for any $t \geq t_3$,

$$[u(t - r(t))]^+ \leq E\varepsilon + \sigma \in \Omega_2. \qquad (16)$$

Since $B = \operatorname{diag}\{b_i\}$ and $b_i > 0$, $i = 1, 2, \ldots, n$, for the above $\varepsilon > 0$ and x^*, there must be a $T > 0$ such that

$$\int_T^\infty e^{-Bs} \Big([A]^+ H(x^*) + P \Big) ds \leq \varepsilon E. \qquad (17)$$

So, from (10), (16) and (17), when $t \geq t_3 + T$, we obtain

$$\begin{aligned} [u(t)]^+ &\leq e^{-B(t-t_0)} [u(t_0)]^+ + \int_{t_0}^t e^{-B(t-s)} \Big([A]^+ H([u(s)]_r^+) + P \Big) ds \\ &\leq e^{-B(t-t_0)} [\phi]_r^+ \\ &\quad + \Big\{ \int_{t_0}^{t-T} + \int_{t-T}^t \Big\} e^{-B(t-s)} \Big([A]^+ H([u(s)]_r^+) + P \Big) ds \end{aligned}$$

$$\begin{aligned}
&\le e^{-B(t-t_0)}[\phi]_r^+ + \int_T^\infty e^{-Bs}\Big([A]^+H(x^*)+P\Big)ds \\
&\quad + \int_{t-T}^t e^{-B(t-s)}\Big([A]^+H(E\varepsilon+\sigma)+P\Big)ds \\
&\le e^{-B(t-t_0)}[\phi]_r^+ + \varepsilon E + (I-e^{BT})B^{-1}\Big([A]^+H(E\varepsilon+\sigma)+P\Big) \\
&\le e^{-B(t-t_0)}[\phi]_r^+ + \varepsilon E + B^{-1}\Big([A]^+H(E\varepsilon+\sigma)+P\Big). \qquad (18)
\end{aligned}$$

Combining (18) with the definition of superior limit again, there are $t_k \ge t_3 + T$, $k = 1, 2, \ldots$ such that $\lim_{k\to+\infty} u(t_k) = \sigma$, and

$$[u(t_k)]^+ \le e^{-B(t_k-t_0)}[\phi]_r^+ + \varepsilon E + B^{-1}([A]^+H(E\varepsilon+\sigma)+P). \qquad (19)$$

Letting $t_k \to +\infty$, $\varepsilon \to 0$, we have

$$\sigma \le B^{-1}([A]^+H(\sigma)+P) = \Delta(\sigma) + \sigma. \qquad (20)$$

This implies that $\Delta(\sigma) \ge 0$, which contradicts that $\sigma \in \Omega_2$. Hence, we obtain $\sigma \in \Omega_1 - \Omega_2$, and the proof is completed.

Corollary 1. In addition to the conditions of Theorem 2, if $\Omega_1 - \Omega_2 = \{0\}$, then $u(t) \equiv 0$ is the equilibrium point of system (1) and it is asymptotically stable, and $D_2 = \{\phi \in C_\Omega | [\phi]_r^+ \in \Omega_1\}$ is its attraction domain.

Theorem 3. If (4) with $h_j(z) = h_j z$ ($h_j \ge 0$ in constant) holds and $\rho(c) < 1$, where $\rho(c)$ denotes the spectral radius of the matrix $c = B^{-1}[A]^+H$ and H =diag$\{h_i\}$, then $D = \{\phi \in C | [\phi]_r^+ \le K, K = [B(I-c)]^{-1}P\}$ is a global attracting set of (1).

Proof. Since $h_j(z) = h_j z$, we have $H(x) = Hx$ and

$$\Delta(x) = B^{-1}([A]^+Hx + P) - x = (c-I)x + B^{-1}P.$$

From Theorem 9.16 in [22] and the fact that $\rho(c) < 1$, there is a vector $W_0 \ge 0$ such that $(I-c)W_0 > 0$. For a given $\phi \in C$, there exists a sufficiently large number β such that

$$[\phi]_r^+ < x^* \triangleq \beta W_0 \quad \text{and} \quad (I-c)x^* > B^{-1}P, \text{ i.e., } \Delta(x^*) < 0. \qquad (21)$$

By Theorems 1 and 2, the set $D = \{\phi \in C | [\phi]_r^+ \leq x^*\}$ is positive invariant and attracting set of (1). This implies that there is a constant $\sigma \geq 0$ such that

$$\overline{\lim}_{m\to\infty}[u(t)]^+ = \sigma. \tag{22}$$

Imitating the proof of (20), we have

$$\sigma \leq c\sigma + B^{-1}P. \tag{23}$$

Again using Theorem 9.16 in [22], we have $(I - c)^{-1} \geq 0$ and

$$\sigma \leq (I - c)^{-1}B^{-1}P = [B(I - c)]^{-1}P. \tag{24}$$

This implies that D attracts the solution $x(t, t_0, \phi)$ through (t_0, ϕ). From the arbitrariness of $\phi \in C$, the proof is completed.

Multiplying both sides of (2) by $u^T(t)$, we have

$$\frac{d}{dt}\|u(t)\|_2^2 = -2u^T(t)Bu(t) + 2u^T(t)[Ag(u(t - r(t))) + P(t)], \tag{25}$$

where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^n$. Let $\|u(t)\|_{2r} = \sup_{-r\leq s\leq 0}\|u(t+s)\|_2$, $b = \min_{1\leq i\leq n}\{b_i\}$ and M =min$\{\|A\|_2\|H\|_2, \|[A]^+H\|_2\}$. Then

$$\frac{d}{dt}\|u(t)\|_2^2 \leq -2b\|u(t)\|_2^2 + 2M\|u(t)\|_{2r}^2 + 2\|P\|_2\|u(t)\|_{2r}, \tag{26}$$

which is similar to the first inequality of (11) (or (18)) with $[u]^+ = \|u\|_2^2$, $B = 2b$ and $[A]^+H([u]_r^+) = 2M\|u(t)\|_{2r}^2 + 2\|P\|_2\|u(t)\|_{2r}$. Following the proofs of Theorems 1 and 2, it is easy to derive the following theorem.

Theorem 4. (4) with $h_j(z) = h_j z$ holds and

$$b > M = \min\{\|A\|_2\|H\|_2, \|[A]^+H\|_2\},$$

then $D = \{\phi \in C | [\phi]_{2r}^+ \leq K, K = \frac{\|P\|_2}{b-M}\}$ is a global attracting set of (1).

In Theorems 3 and 4, when $P = 0$, $\{0\}$ is an equilibrium point of system (1), then we have the following corollary.

Corollary 2. If (4) with $h_j(z) = h_j z$ holds and $P = 0$, then the equilibrium point $u(t) \equiv 0$ of (1) is globally asymptotically stable (GAS) if

$$b > M = \min\{\|A\|_2\|H\|_2, \|[A]^+H\|_2\} \tag{27}$$

or

$$\rho(c) < 1, \quad \text{where } c = B^{-1}[A]^+H. \tag{28}$$

Remark 1. van den Driessche and Zou[1, Theorem 2.4] showed that system (1) has a GAS equilibrium point if g is bounded, (4) with $h_j(z) = h_j z$ holds, and

$$\frac{\lambda}{b}\|A\|_2 < 1 \quad (\text{where } \lambda = \max_{1\le i\le n} h_i), \tag{29}$$

which implies that $b > \lambda\|A\|_2 = \|A\|_2\|H\|_2 \ge M$, that is, (27) holds. Therefore, we obtain the same result as Theorem 2.1 in [1] under weaker conditions.

Condition (28) gives the main result in [9, Theorem 5].

4. Examples

Example 1. Consider the system:

$$\dot{x}(t) = -8x(t) + x^2(t-3)\sin(x(t-3)) + 12\sin t. \tag{30}$$

Obviously, $b = 8$, $H(\|x(t)\|_2) = \|x(t)\|_2^2, P = 12$, and $\Delta(z) = \frac{1}{8}z^2 - z + \frac{3}{2} < 0$ for $2 < z < 6$. So by Theorems 1 and 2, we have $D_1 = \{\phi \in C | \|\phi\|_3 \le 2\}$ is an attracting set, and $D_2 = \{\phi | \|\phi\|_3 < 6\}$ is the attraction basin of D_1.

Example 2. Consider the system:

$$\begin{cases} \dot{u}_1(t) = -4u_1(t) + g_1(u(t-r)) + p_1(t) \\ \dot{u}_2(t) = -8u_2(t) + g_2(u(t-r)) + p_2(t), \end{cases} \tag{31}$$

where g_i, and p_i $(i = 1, 2)$ are continuous functions, and satisfy $|p_1| \le 7/2, |p_2| \le 7$ and

$$\|g_1\|_r \le (\|u_1\|_r^2 + \|u_2\|_r^2)/2, \quad \|g_2\|_r \le \|u_1\|_r^2 + \|u_2\|_r^2.$$

It is obvious that $B = \mathrm{diag}\{4, 8\}$, $A = I$ and

$$H(z) = \begin{pmatrix} \frac{1}{2}\left(z_1^2 + z_2^2\right) \\ z_1^2 + z_2^2 \end{pmatrix}, P = \begin{pmatrix} \frac{7}{2} \\ 7 \end{pmatrix}.$$

By elementary calculations, we have

$$\Delta(z) = \begin{pmatrix} \frac{1}{8}(z_1^2 + z_2^2) + \frac{7}{8} - z_1 \\ \frac{1}{8}(z_1^2 + z_2^2) + \frac{7}{8} - z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}[(z_1 - 4)^2 + z_2^2] - \frac{9}{8} \\ \frac{1}{8}[z_1^2 + (z_2 - 4)^2] - \frac{9}{8} \end{pmatrix}.$$

It follows from $\Delta(z) = 0$ that $z_1 = z_2$ and $2z_1^2 - 8z_1 + 7 = 0$, that is,

$$z_1 = z_2 = 2 \pm \frac{\sqrt{2}}{2}.$$

Therefore, we obtain that

$$\begin{aligned} \Omega^* &= \{z \in \mathbb{R}_+^2 | (z_1 - 4)^2 + z_2^2 < 9 \bigcap z_1^2 + (z_2 - 4)^2 < 9\}, \\ \Omega_1 &= \{z \in \mathbb{R}_+^2 | z_i < 2 + \frac{\sqrt{2}}{2},\ i = 1, 2\}, \\ \Omega_2 &= \{z \in \mathbb{R}_+^2 | (z_1 - 4)^2 + z_2^2 < 9 \bigcup z_1^2 + (z_2 - 4)^2 < 9\}. \end{aligned}$$

By Theorems 1 and 2, we know that the set

$$D_1 = \{\phi \in C | \|\phi_1\|_r \le 4 - \sqrt{9 - \|\phi_2\|_r} \text{ and } \|\phi_2\|_r \le 4 - \sqrt{9 - \|\phi_1\|_r}\}$$

is an attracting set of system (31) and

$$D_2 = \left\{\phi \in C | [\phi]_r^+ < \left(2 + \frac{\sqrt{2}}{2}\right)[1, 1]^T\right\}$$

is the attraction basin of D_1.

Remark 2. It is obvious that

$$D_1 \subset D_1^* \triangleq \left\{\phi \in C | [\phi]_r^+ < \left(2 - \frac{\sqrt{2}}{2}\right)[1, 1]^T\right\}.$$

For the sake of simplicity, we may say D_1^* is an attracting set of D_2.

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